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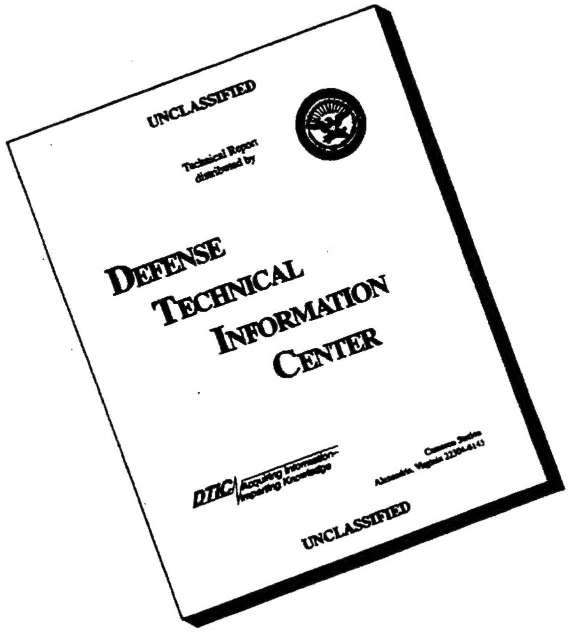

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PURDUE UNIVERSITY

STRUCTURAL SYSTEMS AND

CONTROL LABORATORY

NEW SPACE STRUCTURE AND CONTROL DESIGN CONCEPTS

R. E. Skelton, Director
SSCL

FINAL REPORT

8/96

AFOSR Grant F49620-92-J-0202

New Space Structure and Control Design Concepts

Executive Summary

The objective of this research is to develop a technique to integrate structure and control design with applications to the Phillips Lab Laser Crosslink Structure (LCS). This structure acts as a mounting frame for a proposed laser-based communications system attached to a DSP surveillance satellite; the laser system is part of a planned upgrade from the radio-based system currently employed. Since the laser-based communications system is about ten times more sensitive to transmitter/receiver line-of-sight errors than the current communications system, an active vibration control system is necessary. The goal of this investigation is to redesign the LCS and an appropriate controller simultaneously so that the vibration imparted to the structure by the satellite is compensated for by the control system, thus facilitating communications. The results may be summarized as follows:

1. A finite element model of the LCS was developed for the Phillips TRW design. The freedom in this model includes mass density of the composite tubes, modulus of elasticity, cross sectional area (representing the number of wraps of the composite material), and the length of each finite element. A new finite element is developed to incorporate induced strain actuation for any given beam and piezoelectric cross-section geometries. The controller design freedom is the variance upper bounds allowed in the pointing errors for the laser, in the presence of noisy disturbances from the spacecraft.

2. An algorithm is developed to optimally choose the parameters in both the structure and the controller. The algorithm is guaranteed to converge to the optimal solution. This is believed to be the first solution to a nonlinear programming problem integrating structure and control designs. Past approaches have only given necessary conditions.

3. The results of the LCS redesign when compared to the original Phillips/TRW design for the same variance performances:

   - savings of 22% mass of composite structure
   - savings of 32% control energy
   - the mass is not directly penalized in the design, but it is indirectly penalized by minimizing control energy (for a given performance constraint, it is easier to push around small mass)

4. Software is developed to solve this generic problem which we call Optimal Mix of Passive and Active Control (OMPAC).

5. The set of all stabilizing combinations of structural parameters, piezoelectric mechanical and electrical properties, and control parameters have been characterized. For the LCS structure this allows the parametrization of stabilizing control gains as a function of the composite tube cross-sectional areas, the location and dimensions of the piezoelectric actuators, and the electrical/mechanical properties of the actuator.

6. Covariance controllers are derived for those applications when desired covariances are known, or when upperbounds on the output covariance matrix is specified.

7. These results were very helpful as a starting point for other DOD research to extend the optimization beyond the covariance performance criteria of this grant (F49620-92-0202) to include H_2/H_∞ criteria.

In conclusion, this research has taken a giant step toward the creation of a scientific procedure for the DESIGN process where the space structure and the controller are optimized together as a system.

TABLE OF CONTENTS

A. Mathematical Models of Laser Crosslink Structure

A. Mathematical Models of Laser Crosslink Structure

Introduction:

The LCS is a support structure, connected to a spacecraft, and designed to support a laser weapon. It is required that the structure be actively controlled so as to attenuate the spacecraft-induced disturbances to assure the highest pointing accuracy possible for the laser beam.

Design Requirements:

The LCS control problem is to guarantee an absolute bound on the line of sight pointing error in the presence of a class of uncertain disturbances having a known energy bound. The free design parameters are the material properties of the truss members and the dynamics of the controller.

Objective

Simultaneously choose the structural the actuator, the sensor and the control design parameters to minimize the active control energy required to accomplish the overall system design requirements. One of the contributions of this research is to develop a *theory for design*, deducing design requirements for each *component* of a system, given overall *system* performance requirements. Finally a workstation environment for rapid design is sought.

I. Finite Element Modeling of the Laser Crosslink Structure

We have developed a finite element model of the laser crosslink structure (LCS) to be built by Phillips Labs, using only the six structure members which significantly influence the motion of the laser platform. Figure 1 shows a view of the LCS and it also depicts the crosslink structure idealization used to develop finite element models of the LCS. We have made several assumptions in order to develop our LCS finite element model, and these assumptions are as follows:

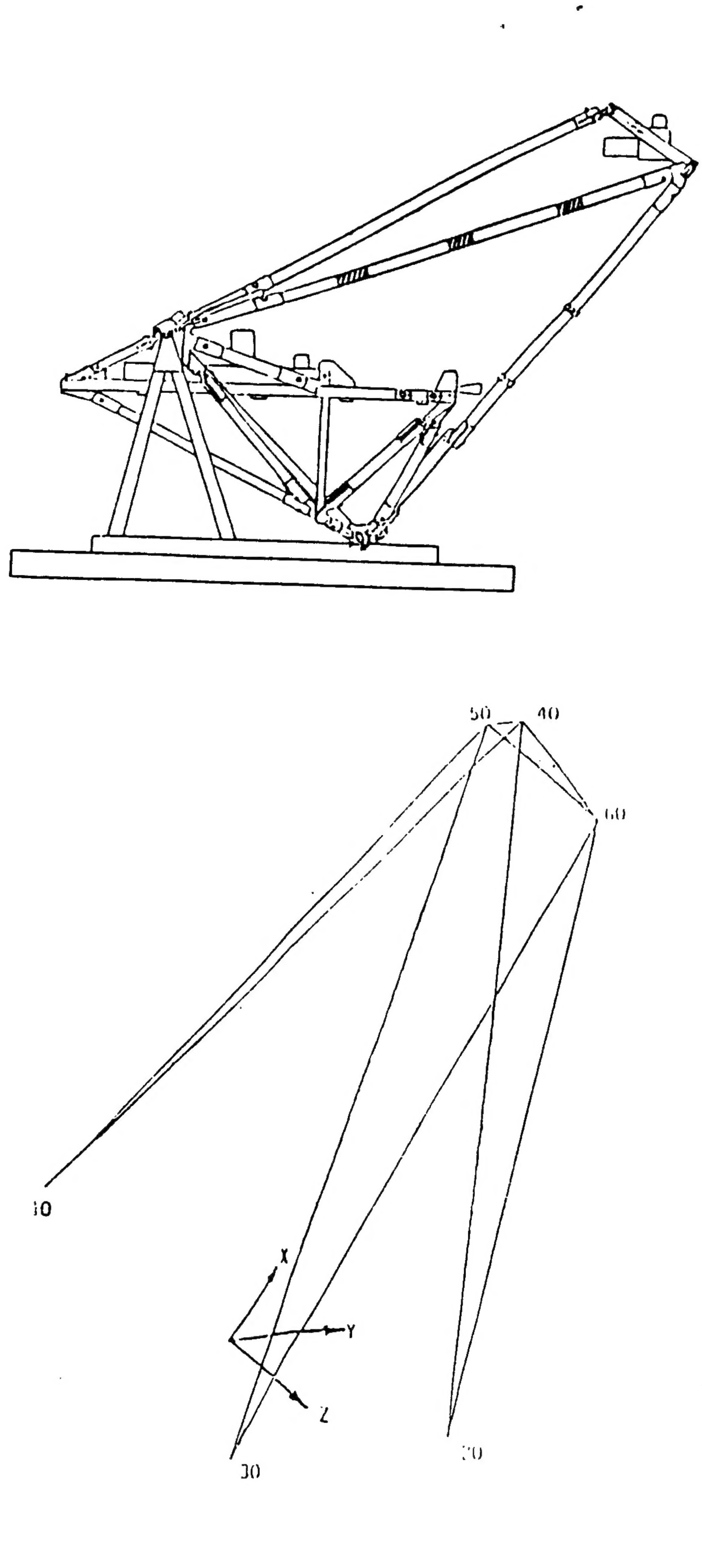

Figure 1

(1) The LCS can be modeled as a truss.

(2) Each structure member is uniform in composition and shape. This assumption excluded modeling the effects of the end sleeves and fitting blades to the structure dynamics.

(3) Mechanical damping is a function of the mass and stiffness matrices of the FEM, i. e., we assume Rayleigh damping.

Although we have written software to assemble the mass and stiffness matrices of an LCS finite element model, we have yet to include in our model the dynamics of piezo-electric materials we assume will be imbedded into the structure members to act as actuating devices. See Appendix A for more information about the LCS.

Our investigation of the LCS finite element modeling will consider three key modeling parameters that will affect the control system design and the LCS redesign: the number of finite elements used to model each structure member, the type of finite element, and the length of each finite element.

II. Integrating Structure and Control Design: An Example

To illustrate our procedure, we begin with a single truss member. We shall design the member so that the design helps to minimize the active control energy required to control the dynamics of the structure to meet required performance objectives. In this approach we consider two broad issues which contribute to the effectiveness of a control design to meet the specified objectives: the design and modeling of the structure.

Truss Member Design

The choice of the material properties and the truss member geometry determine the density, elasticity, cross-sectional area, and length of each truss member. When we change any of these parameters to modify truss dynamics performance, the process becomes a part of the design issue.

TRUSS MEMBER DESIGN PARAMETERS

DENSITY	ρ
ELASTICITY	E
CROSS-SECTION AREA	A
LENGTH OF THE MEMBER	L

Note that a truss member need not be uniform along its entire length, so that there may be sub-members to consider.

Truss Modeling

If we assume that the truss dynamics are modeled via finite element methods, the choice of the number of finite elements modeling each member, N, the type of finite elements (denoted by the degree of the polynomial describing axial displacement), n, and the lengths of each finite element, Lj, are considerations in the modeling process. When we change any of these parameters to modify the fidelity of the truss model, the process becomes part of the modeling issue.

TRUSS MEMBER MODELING PARAMETERS

NUMBER OF FINITE ELEMENTS PER MEMBER	N
TYPE OF ELEMENT	n
LENGTH OF EACH FINITE ELEMENT	L_j

In order to determine how both the design parameters and the modeling parameters affect the controller design issues, we considered a uniform, tubular LCS truss member depicted by Figure 2. In this example one end of the member is inertially fixed. The structure was controlled with minimum energy, assuming that the structure is subject to axial control forces and the displacements and velocities at each of the modeled nodes $\{u_1(t), u_2(t), u_3(t)\}$ are the only measurements available to the control system, and it was desired to keep these displacements within specified values. The control force, $\mathcal{F}_c(t)$, was applied to the free end of the member. In the development of the finite element model of the member, we limited the scope of this example by using three linear finite elements to model the member dynamics. Thus the mass and stiffness matrices for the three finite elements are

$$M_j \equiv \frac{\rho_j A_j L_j}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad K_j \equiv \frac{E_j A_j}{L_j}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad j = 1,2,3 \ .$$

Rayleigh damping was assumed to capture the dependence of damping on the member mass and stiffness matrices.

Table 1 gives the values for the physical properties of each finite element of the truss member.

Table 1

Physical properties of the member

OUTER DIAMETER (inches)	1.050
THICKNESS (inches)	0.050
CROSS-SECTIONAL AREA (in^2)	0.0805
ELASTICITY (psi)	16.0×10^6
DENSITY ($slugs/in^3$)	1.3469×10^{-3}

Rayleigh damping coefficients

$$D \equiv \alpha M + \beta K$$

α	0.001
β	1.0e-07

One way to integrate the design and modeling problems is to define the following design parameters for this example:

$$p_j \equiv \rho_j A_j L_j, \quad j = 1,2,3$$

$$q_j \equiv \frac{E_j A_j}{L_j}, \quad j = 1,2,3$$

where the sum of the design parameters L_1, L_2, L_3 is fixed,

$$L = L_1 + L_2 + L_3 \quad .$$

The redesign algorithm begins with an FEM for the given design,

$$\mathcal{M}\ddot{u}(t) + (\alpha\mathcal{M} + \beta\mathcal{K})\dot{u}(t) + \mathcal{K}u(t) = \Phi\mathcal{F}_c\ ,$$

$$u(t) \equiv \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

and a control law

$$\mathcal{F}_c = G\begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix} \qquad (1)$$

which meets the output variance constraints (OVC):

$$\int_0^\infty u_j^2(\tau)d\tau \le \sigma_j^2\ ,\ \ j = 1,2,3\ .$$

The OVC constraints used in this example are given in Table 2.

Table 2

OVC Bounds for the truss redesign member example

σ_1^2	4.7690e-08
σ_2^2	2.5888e-07
σ_3^2	2.7354e-07

The redesign algorithm computes optimal values for the design parameters $\{p_1, p_2, p_3, q_1, q_2, q_3\}$ and simultaneously computes a control gain which minimizes a quadratic cost function, subject to the constraint that the closed-loop system matrix is preserved and the redesigned stiffness matrix is bounded by

$$\underline{\mathcal{K}} \le \text{trace}[\mathcal{K} + \Delta\mathcal{K}] \le \overline{\mathcal{K}}\ .$$

This last constraint is imposed to place a lower bound on the stiffness matrix of the

redesigned member. The redesigned FEM dynamics are represented by the following:

$$(\mathcal{M} + \Delta\mathcal{M})\ddot{u}(t) + (\alpha\mathcal{M} + \beta\mathcal{K} + \alpha\Delta\mathcal{M} + \beta\Delta\mathcal{K})\dot{u}(t) + (\mathcal{K} + \Delta\mathcal{K})u(t) = \Phi\mathcal{F}_c(t)$$

$$\Delta\mathcal{M} \equiv \mathcal{P}_1\delta p_1 + \mathcal{P}_2\delta p_2 + \mathcal{P}_3\delta p_3$$

$$\Delta\mathcal{K} \equiv Q_1\delta q_1 + Q_2\delta q_2 + Q_3\delta q_3$$

$$\mathcal{F}_c(t) = \tilde{G}\begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix} .$$

The matrix G in (1) is determined to solve the OVC control problem. Then, the parameters δp_1, δp_2, $i = 1,2,3$, and $\tilde{G}$ are determined to *match* the closed loop system matrix (all eigenvalues, eigenvectors), by a *quadratic programming* algorithm, with guaranteed stability and convergence to a global optimum. See Appendices B, C and D.

Numerical Results

The finite element model of the truss member prior to member redesign is characterized by its mass, stiffness, and damping matrices:

$$\mathcal{M} = \begin{bmatrix} 1.6264e{-}03 & 5.4214e{-}04 & 0 \\ 5.4214e{-}04 & 1.8121e{-}03 & 1.1645e{-}04 \\ 0 & 1.1645e{-}04 & 4.6581e{-}04 \end{bmatrix} , \quad \mathcal{K} = \begin{bmatrix} 1.5457e{+}06 & -5.1522e{+}05 & 0 \\ -5.1522e{+}05 & 3.0737e{+}06 & -2.5585e{+}06 \\ 0 & -2.5585e{+}06 & 2.5585e{+}06 \end{bmatrix}$$

$$\mathcal{D} = \begin{bmatrix} 1.5457e{-}01 & -5.1522e{-}02 & 0 \\ -5.1522e{-}02 & 3.0738e{-}01 & -2.5585e{-}01 \\ 0 & -2.5585e{-}01 & 2.5585e{-}01 \end{bmatrix} .$$

Our redesign algorithm resulted in the following redesign FEM which is optimal with respect to the constraints we have imposed:

$$\mathcal{M} + \Delta\mathcal{M} = \begin{bmatrix} 1.4728e{-}03 & 4.9094e{-}04 & 0 \\ 4.9094e{-}04 & 1.6410e{-}03 & 1.0545e{-}04 \\ 0 & 1.0545e{-}04 & 4.2182e{-}04 \end{bmatrix} , \quad \mathcal{K} + \Delta\mathcal{K} = \begin{bmatrix} 1.3997e{+}06 & -4.6656e{+}05 & 0 \\ -4.6656e{+}05 & 2.7834e{+}06 & -2.3169e{+}06 \\ 0 & -2.3169e{+}06 & 2.3169e{+}06 \end{bmatrix}$$

$$\mathcal{D} + \Delta\mathcal{D} = \begin{bmatrix} 1.3997\text{e}{-01} & -4.6656\text{e}{-02} & 0 \\ -4.6656\text{e}{-02} & 2.7835\text{e}{-01} & -2.3169\text{e}{-01} \\ 0 & -2.3169\text{e}{-01} & 2.3169\text{e}{-01} \end{bmatrix},$$

$$\tilde{G} = \begin{bmatrix} 34.9902, & -19.4386, & -13.4445, & -0.0533, & -0.3006, & -0.1009 \end{bmatrix}$$

where the optimal values for the design parameters are given in Table 3. For this example we took

$$5.4 \times 10^5 \text{ lb/in} \le \text{trace}[\mathcal{K} + \Delta\mathcal{K}] \le \infty .$$

It is easy to see that both the mass and stiffness matrices have decreased, but what is not obvious is that the redesign preserved the modal frequencies and modal damping (by the Rayleigh assumption) of the open loop system, i.e., the following was preserved:

$$\mathcal{M}^{-1}\mathcal{K} = (\mathcal{M} + \Delta\mathcal{M})^{-1}(\mathcal{K} + \Delta\mathcal{K}) \ .$$

Since the closed-loop system matrix before and after the redesign was constrained to be the same, note that together with the above equality the redesign constraints imply that

$$\mathcal{M}^{-1}\Phi G = (\mathcal{M} + \Delta\mathcal{M})^{-1}\Phi\tilde{G} .$$

Hence we see that the redesign preserved the product of the control distribution matrix Φ and the controller gain, so that a decrease in truss member mass implied a decrease in required control gain, which is equivalent to a reduction of control effort in the redesign method.

Table 3

Values for the optimal design parameters

$p_1 \equiv \rho A L_1$	2.2659e-03
$p_2 \equiv \rho A L_2$	4.5318e-03
$p_3 \equiv \rho A(L - L_1 - L_2)$	1.2168e-03
$q_1 \equiv EA/L_1$	1.4356e+06
$q_2 \equiv EA/L_2$	7.1779e+05
$q_3 \equiv EA/(L - L_1 - L_2)$	2.6733e+06

We considered the case where the redesigned member was uniform over its length. This led to necessary conditions on the design parameters to make them realizable from physical parameters $\{\rho, E, A, L_1, L_2, L_3\}$ which are given by:

$$p_1 q_1 = p_2 q_2 = p_3 q_3 \ .$$

If these conditions are met, then the element lengths L_1, L_2, and L_3 must sum to the total length of the truss member, L, and are uniquely determined. Also, the design parameters must satisfy:

$$\frac{E}{\rho} = \frac{q_1}{p_1} L_1^2 = \frac{q_2}{p_2} L_2^2 = \frac{q_3}{p_3} (L - L_1 - L_2)^2 \ .$$

For the optimal values of the design parameters listed in Table 3, it may be shown that the design freedom is to choose E and ρ subject to

$$\frac{E}{\rho} = \frac{q_1}{p_1} L_1^2 = 1.4255e{+}11 \ .$$

For the truss member redesign, once any one of $\{\rho, E, A\}$ is chosen the remaining quantities are determined by the design parameters $\{p_1, p_2, p_3, q_1, q_2, q_3\}$. Therefore we computed ρ, A, and E for two cases: (1) E equal to the original design value and (2) A equal to the original design value. Table 4 summarizes the truss member redesign. Note that the quantity Er-1 is constant before and after the redesign, so that the ratio is

preserved by the design method, i.e.,

$$\frac{E}{\rho} = \frac{q_1}{p_1} L_1^2 = \frac{q_2}{p_2} L_2^2 = \frac{q_3}{p_3} (L - L_1 - L_2)^2$$

is an invariant in truss members using linear elements.

Table 4

Physical properties of the member redesign

	Original Design	E fixed	A fixed
ρ (slugs/in^3)	1.3469e-03	*	1.2197e-03
E (psi)	16.0 x 10^6	*	14.49 x 10^6
A (in^2)	0.0805	7.2900e-02	*
L_1 (in)	15.0	15.0	15.0
L_2 (in)	30.0	30.0	30.0
L_3 (in)	8.055	8.055	8.055

Tables 5a-5b summarize the results of the truss member and controller redesign. Note that the redesign preserved the OVC constraints for given initial conditions $\{u_1(0), u_2(0), u_3(0), \dot{u}_1(0), \dot{u}_2(0), \dot{u}_3(0)\}$ as guaranteed by our theory.

* These values were chosen to be the same as the original design.

Table 5a

Output variances due to initial conditions

OVC Bound	Initial FEM	FEM Redesign
7.6304e-07	7.6304e-07	7.6304e-07
4.1421e-06	2.9139e-06	2.9139e-06
4.3767e-06	3.2784e-06	3.2784e-06

Table 5b

Active control effort for the FEM

Initial FEM	FEM Redesign	Change (%)
7.4913e+01	6.1431e+01	-18.0

It is important to realize that in the most general design problem for this example we have at our disposal eleven design and model parameters, these being $\{\rho_1,\rho_2,\rho_3,E_1,E_2,E_2,A_1,A_2,A_3,L_1,L_2\}$, while this example used only *five* of the modeling and the control design parameters. The choice of different material properties ρ_i, E_i will be utilized in the modeling of the piezoelectric material forming the composite structure for the next stage of development to integrate structure, actuator, sensor, and control design. This will allow the structures to *truly* be "smart".

III. Conclusions to date

- A convergent algorithm has been developed which generates optimal choices for material selection and control design for the Laser Crosslink Structure (LCS), or any other similar truss structure.

- Software is now available for this algorithm.

- Preliminary results have produced an invariant combination of the material properties for truss elements. We hope to show that such invariant properties will apply to designs of other classes of systems.

Apart from the theoretical issues explored in the example of section II, other theoretical issues which remain open are

How to optimally redesign a vector second order system with white noise disturbances

How to redesign a vector second order system or linear system with parameter dependent control and output matrices.

How to formulate mathematically tractable constraints on the design parameters $\{p_j, q_j\}$, which are directly related to dynamic and static loads on a structure.

The development of a covariance control theory for vector second order systems.

Acknowledgement:

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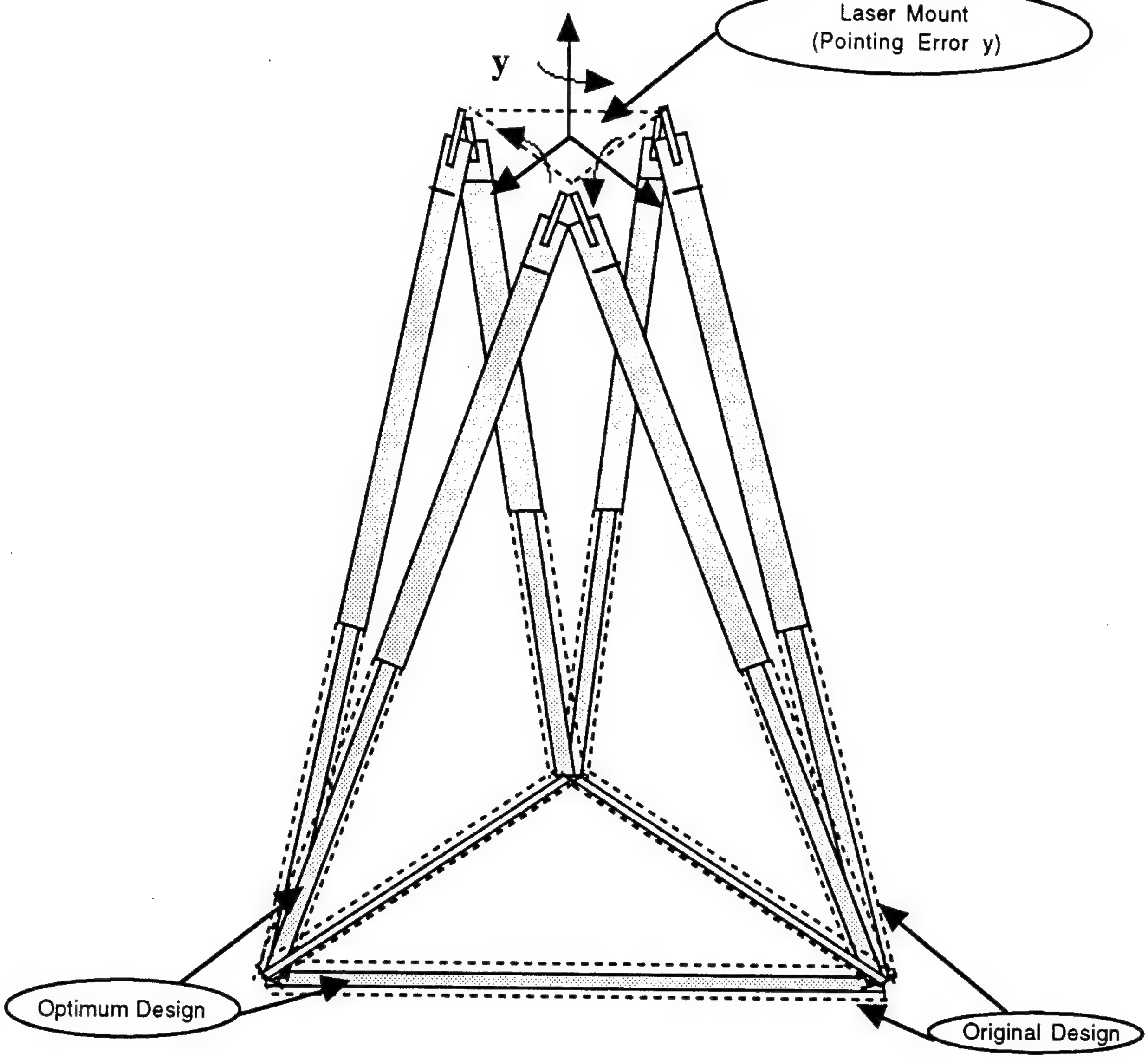
PHILLIPS LAB LASER CROSSLINK STRUCTURE



CONCLUSIONS:

MONOTONIC CONVERGENCE (TO WITHIN 10^{-18} ERROR) IN 8 ITERATIONS.

REDESIGN GUARANTEES SPECIFIED POINTING PERFORMANCE BOUNDS $\|y\| \leq \sigma$ OF LASER.

ACTIVE CONTROL REDUCTION 13%.

STRUCTURAL MASS REDUCTION: 8%.

***SOFTER* STRUCTURE EASIER TO CONTROL!**

APPENDIX A

Physical Data for the LCS members

OUTER DIAMETER (inches)	1.050
THICKNESS (inches)	0.050
ELASTICITY (psi)	16.0×10^6
DENSITY (slugs/in^3)	1.3469×10^{-3}

Coordinate Locations of the Pin Joints

NodeID	X-coordinate	Y-coordinate	Z-coordinate
10	0.0000	0.0000	0.0000
20	27.2315	19.5420	0.0000
30	27.2315	-19.5420	0.0000
40	6.2625	3.0310	45.7960
50	6.2625	-3.0310	45.7960
60	15.6005	0.0000	44.8420

Modal Cost Analysis

The modal cost analysis of an LCS finite element model using third order finite elements to model the axial dynamics is given in Table A.1. The purpose of modal cost analysis is to calculate the contribution of the i^{th} modal coordinate to a quadratic cost function. For this analysis we took as our output vector, y(t), the relative translation of each node of the laser platform to the remaining platform nodes. The control actuators were idealized point-actuators applied at the center of each of the six structure members which attach to the laser platform. We assumed that the intensity of the impulses applied to the control channels was unity, and we weighted the components of the output vector equally by choosing Q to be identity.

Table A.1 summarizes the results of the modal cost analysis, sorting the modes with largest contribution to the modal cost to the modes with smallest contribution. Note

that the contribution to the modal cost does not increase monotonically with mode number.

Table A.1

Summary of a modal most analysis of the LCS model

Mode	Mode	Modal Cost
6	1.9585e+03	8.2981e-05
4	1.8844e+03	6.0630e-05
5	1.9399e+03	2.4989e-05
1	9.7471e+02	1.4526e-05
2	1.3461e+03	6.5163e-06
12	7.0234e+03	1.3939e-06
10	6.7798e+03	1.0512e-06
11	6.9992e+03	4.4241e-07
7	4.0901e+03	1.7389e-07
3	1.6946e+03	8.8014e-08
8	4.5216e+03	3.9574e-09
9	5.5007e+03	3.4057e-10
14	1.7316e+05	2.4851e-14
13	1.1643e+05	2.0676e-14
16	2.9294e+05	8.3240e-16
15	2.7001e+05	5.2107e-16
17	4.4357e+05	4.1507e-17
18	5.9451e+05	3.8897e-17

B. Integrated Structure and Controller

INTEGRATED STRUCTURE AND CONTROLLER DESIGN

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Abstract

Modern problems which require severe constraints on the dynamic response will demand a theory for system design, integrating plant design with controller design. Successive covariance approximation is proposed for integrating structure and control designs.

1. Introduction

This paper suggests a need to develop a *theory of design* for dynamic systems, subject to a specific set of performance criteria. The objective is to integrate the different steps in the design process of controlled engineering systems. In the class of problems to be treated, the system is subject to uncertain but bounded errors in modeling, to uncertain but bounded disturbances, and we assume freedom in the choice of some parameters that characterize the properties of each component of the total system. For example, the design procedure for the design of a controlled structure could include: choice of the material, choice of the sensors and actuators (dynamic range, type, signal to noise ratio, location, number) choice of the controller hardware/software (controller complexity, memory, wordlength, quantization strategy, control energy required). The design criteria includes: bounds on both the static response and the dynamic response, controller simplicity and robust performance (in the presence of a specified set of parameter uncertainties, failures, disturbances), determining *component* design requirements given *system* design requirements, minimal cost subject to a performance constraint. Finally, the design theory must be implementable in a workstation environment for rapid design capability. For our special class of design problems, one could develop a mathematical theory and computational algorithms which guarantee convergence to a feasible solution. This contribution would greatly assist or replace (for this special class of problems) the ad hoc procedures currently used in design.

This talk will present one approach called *Successive Covariance Approximation* (SCA) for integrated design of electromechanical systems. The SCA technique is an iterative design of the electrical/structural parameters and the active control parameters, so that in each step the closed-loop system performance closely approximates a predetermined set of desired performance objectives. The design objectives are expressed in terms of the closed-loop state covariance matrix. The active control design step of the iterative design process utilizes the newly developed techniques for covariance control design using alternating projections. Covariance control provides a parameterization of all stabilizing controllers (hence, the unifying theme here is that *any* controller can be derived as a covariance controller) and recent research by the author has parametrized all stabilizing *combinations* of the free structural parameters and the control parameters. The structural design step solves a convex optimization with respect to the structural parameters. Each step minimizes the distance between the closed-loop covariance matrix and the set of desired covariance matrices. The convergence of this iterative structure/controller design method is guar-

anteed by the theory [Skelton 1989; Wahlberg et al. 1990; Rotea and Prasanth 1994; Gevers 1991; Skelton 1988; Li et al. 1989; Lee et al. 1994; Muske and Rawlings 1993; Furuta et al. 1991].

2. Motivation

For 150 years the classical theories of physics, (understanding the physical phenomena), the classical theories of dynamic modeling, (developing mathematical models, such as finite elements, for the (idealized) phenomena) and the more recent theories of control (developing feedback signals *based upon the model*) have been built as *separate* topics (*separated* by boundaries of convenience). Modern engineering has essentially reached the limits of such thinking. We believe that the next revolution in engineering will come through the systematic *destruction* of these disciplinary boundaries, to allow these (artificially separated) disciplines to *cooperate* to achieve performance *beyond the capability of the isolated approaches*. Indeed, *only in this way can one determine the ultimate capability* of the engineering mission. Before this can happen, however, one must take a more fundamental approach to the design, modeling and control of dynamic systems.

A design methodology for controlled mechanical systems should integrate the following subjects:

1) Plant design: Select materials, material properties, and geometry.
2) Sensor and actuator selection: Select the number and location of sensors and actuators, and their precision (dynamic range, signal to noise ratios), and cost.
3) Control design: Develop a control law to guarantee that performance specifications are satisfied for the closed-loop system.
4) Controller implementation: Determine controller precision and complexity (wordlength, computational time delay) required in the control computer.
5) Manufacturing tolerances: Determine the *component* precision required to guarantee a *system* performance requirement.
6) Economical designs: Find the minimal cost design subject to a specified performance constraint. (precision of components is related to cost).

We shall use the word "design" in this proposal to denote such a unified treatment of these disciplines. A theory of design is presently not available. The ad hoc integration of disciplines currently required in design typically requires a very experienced engineer to accomplish a good design. A theory of design would allow good designs to be obtained by less experienced persons, and such a design theory could be taught in universities.

In the most general setting, this goal of developing a mathematical theory of design is too ambitious. Ad hoc iterations will always be required for the most general problems. However, by restricting the problems we treat to a special class, one can expect a fairly complete solution to the design problem.

Finite Signal-to-Noise Models

Contributions toward a theory of design could be developed using a new class of models, called FSN (Finite-Signal-to-Noise) models, described as follows.

The output of a 5 milliwatt amplifier usually contains less noise than the output of a 5 megawatt amplifier. Even if the noise is white, classical additive models of noisy signals do not capture this effect. Consider that $u(t)$ is the signal uncorrelated with $w(t)$, the noise. The traditional white noise model (for the continuous, scalar case) is

$$\begin{aligned} \mathcal{E}\left[w(t)\right] &= 0\ ,\ \mathcal{E}w(t)u(\tau)=0,\ t\geq\tau \\ \mathcal{E}\left[w(t)w(\tau)\right] &= W\delta(t-\tau) \end{aligned} \tag{1}$$

where W is the constant or time varying noise intensity. Let the variance of the signal $u(t)$ be $U \triangleq \mathcal{E}u^2(t)$. Then the intensity of the noise W is constant with respect to the variance of the signal U. Recently we have developed a new model of linear stochastic processes, using "finite-signal-to-noise" (FSN) models as follows. Assume that $u(t)$ and $w(t)$ are uncorrelated as in (1), but that the intensity W depends linearly on the variance of the signal $u(t)$,

$$W = W_0 + \sigma U\ . \tag{2}$$

We define σ^{-1} as the "signal-to-noise ratio" (SNR). We shall call (1), (2) the finite-signal-to-noise FSN model of a noise source. The standard noise model upon which LQG theory is based has an infinite SNR, where $W = W_0$. Now consider Fig. 1 where six FSN sources are added.

- $w_a(t)$ is the actuator noise whose intensity W_a is linearly related to the variance of the signal part of $u(t)$: $W_a = W_{a1} + \sigma_a U = W_{a1} + \sigma_a^2 G^2 X_c$ where for dynamic controllers X_c is the covariance of the controller state, and G is the controller gain.
- $w_p(t)$ is the noise whose intensity depends on the state covariance: $W_p = W_{p1} + \sigma_p^2 X_p$, X_p = plant state covariance. As an example of this type of noise consider that the noise component of turbulent wind forces on a building is a function of the state variables; such as the torsional displacement of the building relative to the wind velocity vector. For larger torsional angles of attack the laminar flow becomes more turbulent, creating a larger noise in the lift component of the forces.
- $w_s(t)$ is the sensor noise whose intensity is linearly related to the variance of the measured signal $z_p(t) = mx_p(t)$: $W_s = W_{s1} + \sigma_s^2 m_p^2 X_p$ (sensor SNR = σ_s^{-1}).
- $w_z(t)$ is the error in the input computation to the controller. In a digital system this would be the roundoff noise in A/D conversion, $W_z = W_{z1} + \sigma_z^2 m_p^2 X_p$.
- $w_x(t)$ is the error in controller state computation.
- $w_u(t)$ is the noise added by controller output computation, $W_u = W_{u1} + \sigma_c^2 G^2 X_c$. In digital controllers this would be roundoff errors from D/A conversion.

A design theory *must* be able to trade component precision with closed loop performance. Let X, G denote the state covariance and control gain of the closed loop system. We shall call any consistent matrix pair ($X > 0$, G) a solution to the *FSN Covariance Control problem.* The standard *Covariance Control Problem* is defined by the case $\sigma_i = 0$ for $i = a, c, s, z$.

It may be shown that any solution of the FSN covariance control problem has a guaranteed stability margin in the sense that all eigenvalues of the closed loop system lie to the left of $-\frac{1}{2}\sigma_p^2$.

The only extension of this result beyond [Iwasaki and Skelton, 1993; Skelton et al. 1994] is the presence of $\sigma_p > 0$. If all σ's are zero, this is exactly the result of [Iwasaki and Skelton, 1993; Skelton et al. 1994] which parametrizes all stabilizing controllers of order $n_c = n_p$. It is possible with very little additional effort to parametrize all controllers of order $n_c \leq n_p$, for both the continuous and discrete case, but we shall not state those results here. When all σ's are zero, such results can be found in [Skelton et al. 1994].

Alternating projections [Grigoriadis, 1994] can be used to solve convex problems in the space of symmetric matrices as follows. Consider the State Covariance Approximation (SCA) problem of minimizing the covariance error from a desired covariance. For example, the set of all covariances X such that $CXC^* \leq \overline{Y}$ for some specified C and $\overline{Y}$ is a convex set that could describe the "desired set" $\mathcal{X}_d$. Let the set of X satisfying of assignable X (X achievable by some G) be denoted $\mathcal{X}_p$, and the set of X satisfying $X > 0$ be denoted by $\mathcal{X}_{pd}$. The SCA problem can be stated as

$$\min_X ||X - X_d||_F$$

subject to $X \in \mathcal{X}_d \cap \mathcal{X}_p \cap \mathcal{X}_{pd}$. Then the computational problem is to find $X_p \in \mathcal{X}_p \cap \mathcal{X}_d \cap \mathcal{X}_{pd}$. The controller is given in terms of X. The alternating projection algorithm [Grigoriadis, 1994] guarantees to find the intersection of convex sets. If no such X exists the same algorithm finds $X \in \mathcal{X}_p \cap \mathcal{X}_{pd}$ that minimizes $||X_p - X_d||_F$. Other approaches to convex control problems are described in [Iwasaki and Skelton, 1993; Skelton et al., 1994; Grigoriadis, 1994; Boyd et al., 1994; Geromel et al., 1993, Grigoriadis and Skelton, 1994b]. Analytical properties (performance, control design and robustness) of the FSN Covariance Control problem should be studied.

Sensor/Actuator Selection

Most control texts begin with the assumption that sensors and actuators are selected before control design. Actually, the problems of modeling, selecting what variables to measure, selecting what variables to control, and selecting a controller are all interdependent problems, with no available scientific procedure to integrate them. Selecting better models and selecting better sensor/actuator variables can sometimes yield a more drastic improvement in robustness and performance than simply using robust control theory on the original model. (This is not a statement against robust control but one to encourage more effort on modeling).

It is well known that performance might be improved by *deleting* a noisy actuator because the

noise has a direct path to the output, and hence the controller can never make the noisy contribution zero. The contribution of a sensor noise to the plant response *can* be made zero by zeroing this sensor gain (in the controller)). Note that the act of deleting the actuator *cannot* be accomplished by the LQG theory since the controller design cannot delete the noise source, it can only *increase* the control signal to try to reduce the relative effect of the actuator noise. (The Kalman filter gain is related to the ratio of plant (actuator in this case) noise to sensor noise, so an increase in actuator noise will increase the Kalman gain. The control gain remains constant since it is not a function of the noise). Hence, LQG theory increases the controller gain when an actuator noise increases. This might be exactly the *wrong thing* to do from an engineering point of view! Performance might be improved by *deleting* the actuator, rather than *increasing* its control gain [Norris and Skelton, 1989; Chen and Seinfeld, 1975; Ichikawa and Ryan, 1979; Goh and Caughey, 1985; Grigoriadis and Skelton, 1994a].

Suppose the model of the plant and noise sources are accurate. Let the admissible set of actuators be an arbitrarily large number of actuators. The following is an open problem.

- *How many (noisy) actuators yield the best closed loop performance* (maximal accuracy)?

In the case of *sensors* the answer is known to be *the entire admissible set* because the Kalman filter can set the gain to zero if the sensor is too noisy. However in the case of actuators the answer is *not* the entire admissible set. There is an optimal number of actuators for the control of linear stochastic processes, but no available theory reveals how many. (This number is *not* necessarily the minimal number required for controllability).

The conclusion of this section is that a theory is needed to suggest which actuators are helping and which are degrading performance. We call this the sensor/actuator selection problem.

A theory is also needed to determine the precision required (e.g. noise allowed) of each component (sensor, actuator, A/D and D/A converters) of the system to satisfy a given performance constraint. The determination of *component* design specifications, given a *system* performance specification is an important open problem. That is, given a *system* performance constraint $\mathbf{CXC}^* < \overline{\mathbf{Y}}$, determine the required precision (σ_i^{-1}) of each *component* of the system.

Integrated Plant and Controller Design

The set of signal-to-noise ratios σ_i^{-1} of the previous section is only one set of system parameters that may be optimized. In some cases, (early in the design phase of a project) the plant itself may be redesigned for improved performance [Grigoriadis and Skelton, 1994a]. For the structural systems in our mechanical system focus let $\mathbf{p}$ be a vector of free parameters in the structure design

$$\mathcal{M}(\mathbf{p})\ddot{\mathbf{q}} + \mathcal{D}(\mathbf{p})\dot{\mathbf{q}} + \mathcal{K}(\mathbf{p})\mathbf{q} = \mathcal{B}(\mathbf{p})(\mathbf{u} + \mathbf{w})$$

$$\begin{aligned} \mathbf{y} &= \mathcal{P}\mathbf{q} + \mathcal{R}\dot{\mathbf{q}} \\ \mathbf{z} &= \mathbf{M}_p\mathbf{x} + \mathbf{v}\,, \quad \mathbf{x}^* = [\mathbf{q}^*\ \dot{\mathbf{q}}^*] \end{aligned}$$

where the parameters appear multilinearly

$$\begin{aligned} \mathcal{M}(p) &= \sum_i p_i\mathcal{M}_i \\ \mathcal{D}(p) &= \sum_i p_i\mathcal{D}_i \\ \mathcal{K}(p) &= \sum_i p_i\mathcal{K}_i\,. \end{aligned}$$

A convergent algorithm for integrating plant and controller design is proposed as follows,

1. For a given $\mathbf{p}$ solve the standard covariance control problem (e.g. by Alternating Projections), see [Grigoriadis and Skelton, 1994b], to get G.
2. For the given controller G solve the convex problem
$$\min_{p_i} ||\mathbf{X} - \mathbf{X}_d||$$
Repeat for $i = 1, 2, \cdots$
3. Return to 1 until convergence.

Convergence of this algorithm to the global optimal solution is not guaranteed. In a recent application of this algorithm a vibration isolation problem (a space-based laser on a support platform) is solved by reducing the control energy by 32% and the structural mass by 22% compared to the original design by the manufacturer [Grigoriadis and Skelton, 1994a]. This involved selection of the cross-sectional areas of the composite structural members, and the control gains, to keep the L_2 output response feasible.

3. Conclusions

We believe that the next era in controls will be to extend the methods to include plant design, jointly with control design. This paper suggests that convergent algorithms should be sought for solving or approximating the FSN Covariance Control Problem, and the Economical Design Problem. Some progress in this direction would allow component design from a given system performance requirement.

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C. A Convergent Algorithm for the Output Covariance Constraint Control Problem

A Convergent Algorithm for the Output Covariance Constraint Control Problem *

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Abstract

This paper considers the optimal control problem of minimizing control effort subject to multiple performance constraints on output covariance matrices Y_i of the form $Y_i \leq \overline{Y}_i$, where $\overline{Y}_i$ is given. The contributions of this paper are a set of conditions that characterize global optimality, and an iterative algorithm for finding a solution to the optimality conditions. This iterative algorithm is completely described up to a user specified parameter. We show that, under suitable assumptions on problem data, the iterative algorithm converges to a solution of the optimality conditions provided this parameter is properly chosen. Both, discrete and continuous time problems are considered.

1 Introduction

Consider the following linear time-invariant system

$$\begin{aligned} \dot{x}_p(t) &= A_p x_p(t) + B_p u(t) + D_p w_p(t) \\ y_p(t) &= C_p x_p(t) \\ z(t) &= M_p x_p(t) + v(t) \end{aligned} \tag{1.1}$$

where x_p is the state, u the control, w_p represents process noise, and v is the measurement noise. The vector y_p contains all variables whose dynamic responses are of interest. The vector z is a vector of noisy measurements.

Suppose that, to the plant (1.1) we apply a full state feedback stabilizing control law of the form

$$u(t) = G x_p(t) \ , \tag{1.2}$$

or a strictly proper output feedback stabilizing control law given by

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + F z(t) \\ u(t) &= G x_c(t) \ . \end{aligned} \tag{1.3}$$

Then, the resulting closed loop system is

$$\begin{aligned} \dot{x}(t) &= A x(t) + D w(t) \\ y(t) &= \begin{bmatrix} y_p(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} C_y \\ C_u \end{bmatrix} x(t) = C x(t) \ , \end{aligned} \tag{1.4}$$

where for the state feedback case we have $x = x_p$ and $w = w_p$, while for the output feedback case we have $x = [x_p^T \ x_c^T]^T$ and $w = [w_p^T \ v^T]^T$. Moreover, formulae for A, C, and D are easy to obtain from (1.1), and (1.2) or (1.3).

Consider the closed loop system (1.4). Let W_p and V denote positive definite symmetric matrices with dimensions equal to the process noise w_p and measurement vector z, respectively. Define $W = W_p$, if the state feedback controller (1.2) is used in (1.4), or $W = block\ diag[W_p,\ V]$ if (1.3) is used in (1.4). Let X denote the closed loop controllability gramian from the (weighted) disturbance input $W^{-1/2}w$. Since A is stable, X satisfies

$$0 = AX + XA^T + DWD^T \ . \tag{1.5}$$

Partition the performance output y_p in (1.4), into $y_p := [y_1^T, y_2^T, \dots, y_m^T]^T$, where $y_i = C_i x \in \mathcal{R}^{m_i}$ for $i = 1, 2, \dots, m$. In this paper we are interested in finding controllers of the form (1.2) or (1.3) that minimize the (weighted) control energy *trace* $RC_u XC_u^T$ with $R > 0$, and satisfy the constraints

$$Y_i = C_i X C_i^T \leq \overline{Y}_i \ , \ i = 1, 2, \dots, m, \tag{1.6}$$

where $\overline{Y}_i > 0$ $(i = 1, 2, \dots, m)$ are given and X solves (1.5). This problem, that we call the the Output Covariance Constraint (OCC) problem, is defined as follows:

The OCC Problem

Find a static state feedback or full order dynamic output feedback controller for system (1.1) to minimize the OCC cost

$$J_{OCC} = trace\, RC_u XC_u^T \ ; \ R > 0 \tag{1.7}$$

subject to (1.5) and (1.6). □

The OCC problem may be given several interesting interpretations. For instance, assume first that w_p and v are uncorrelated zero mean white noises with intensity matrices $W_p > 0$ and $V > 0$. That is, let $\mathcal{E}$ be an expectation operator, and

$$\begin{aligned} \mathcal{E}[w_p(t)] = 0 \ \ &; \ \ \mathcal{E}[w_p(t) w_p^T(t-\tau)] = W_p \delta(\tau) \ , \\ \mathcal{E}[v(t)] = 0 \ \ &; \ \ \mathcal{E}[v(t) v^T(t-\tau)] = V \delta(\tau) \ . \end{aligned} \tag{1.8}$$

Letting $\mathcal{E}_\infty[\cdot] := lim_{t\to\infty}\mathcal{E}[\cdot]$, and $W = W_p$ for the case of state feedback, or $W = block\ diag$ $[W_p, \ V]$ for the output feedback case, it is easy to see that the OCC is the problem of minimizing $\mathcal{E}_\infty u^T R u$ subject to the output covariance constraints $Y_i := \mathcal{E}_\infty y_i(t) y_i^T(t) \leq \overline{Y}_i$. As is well known, these constraints may be interpreted as constraints on the variance of the performance variables or lower bounds on the residence time (in a given ball around the origin of the output space) of the performance variables [10].

The OCC problem may also be interpreted from a deterministic point of view. To see this, define the $\mathcal{L}_\infty$ and $\mathcal{L}_2$ norms

$$\begin{aligned} \| y_i \|_\infty^2 \ &:= \ \sup_{t\geq 0} \ y_i^T(t) y_i(t) \ ; \\ \| w \|_2^2 \ &:= \ \textstyle\int_0^\infty w^T(t) w(t) dt \ , \end{aligned} \tag{1.9}$$

and define the (weighted) $\mathcal{L}_2$ disturbance set

$$\mathcal{W} := \{ w : R \to R^{n_w} and \ \| W^{-1/2} w \|_2^2 \leq 1 \}, \tag{1.10}$$

where $W > 0$ is a real symmetric matrix. Then, for any $w \in \mathcal{W}$, we have [17, 18]

$$\| y_i \|_\infty^2 \leq \overline{\sigma}[Y_i] \ ; \ i = 1, 2, \dots, m, \tag{1.11}$$

and

$$\| u_i \|_\infty^2 \leq [C_u X C_u^T]_{ii} \ ; \ i = 1, 2, \dots, n_u, \tag{1.12}$$

where n_u is the dimension of u. (Here, $\overline{\sigma}[\cdot]$ denotes the maximum singular value and $[\ ]_{ii}$ is the i-th diagonal entry). Moreover, references [17, 18] show that the bounds in (1.11) and (1.12) are the least upper bounds that hold for an arbitrary signal $w \in \mathcal{W}$.

Thus, if we define $\overline{Y}_i := I_{m_i}\epsilon_i^2$ in (1.6) and $R := diag[r_1, r_2, \ldots, r_{n_u}]$ in (1.7), the OCC problem is the problem of minimizing the (weighted) sum of worst-case peak values on the control signals given by

$$J_{OCC} = \sum_{i=1}^{n_u} r_i \{ \sup_{w \in \mathcal{W}} \|u_i\|_\infty^2 \} \tag{1.13}$$

subject to constraints on the worst-case peak values of the performance variables of the form

$$\sup_{w \in \mathcal{W}} \|y_i\|_\infty^2 \leq \epsilon_i^2 \ , \ \ i = 1, 2, \ldots, m. \tag{1.14}$$

This interpretation is important in applications where hard constraints on responses or actuator signals cannot be ignored; such as space telescope pointing or machine tool control.

Control problems related to the OCC problem defined here have been considered before by several authors. See, for example, [6, 9, 5, 1, 3, 15, 16] for work in multiobjective optimal control with quadratic cost functionals, [13, 14, 4, 19] for the so-called variance constraint control problems, and [12] for the so-called generalized $\mathcal{H}_2$ control problem.

In the above references, one may find two different approaches for solving this class of optimal control problems. The approach based on solving the optimality conditions corresponding to the optimization problem at hand [4, 16, 19], and the approach based on reducing the given problem to a finite dimensional convex optimization problem [1, 3, 12].

In this paper, we follow the approach initiated in [4, 19]. Here, we consider a more general and realistic problem, i.e. the OCC problem, than the one studied in [4, 16, 19], and provide an iterative algorithm for solving the optimality conditions corresponding to this problem. Our main contribution is in the algorithm itself. This iterative algorithm is completely described up to a user specified parameter. We show that the algorithm converges to a solution of the optimality conditions (assuming one exists) provided the user specified parameter is properly chosen. Both, discrete and continuous time problems are considered.

The paper is organized as follows. Section 2 provides optimality conditions for the continuous-time OCC problem in the case of state feedback. These conditions comprise one algebraic Riccati equation and one Lyapunov equation. The Riccati equation has a forcing term depending on a matrix Q (which represents the Kuhn-Tucker multipliers) that must be determined. An algorithm for finding this matrix Q is given, and its convergence analyzed. Section 2 concludes with the extension of the state feedback results to the output feedback case. Section 3 is the discrete-time version of Section 2. An example is presented in Section 4 to illustrate the performance of the algorithm. Section 5 gives the conclusions of this work.

The notation used in this paper is fairly standard. Given the continuous-time algebraic Riccati equation

$$0 = A_p^T K + K A_p - K B_p R^{-1} B_p^T K + C_p^T Q C_p \ ,$$

we say that K is the stabilizing solution if $K = K^T$ satisfies the Riccati equation and $A_p - B_p R^{-1} B_p^T K$ has all eigenvalues in the open left half plane. Similarly, given the discrete-time algebraic Riccati equation

$$K = A_p^T K A_p - A_p^T K B_p (R + B^T K B_p^T)^{-1} B_p^T K A_p + C_p^T Q C_p \ ,$$

we say that K is the stabilizing solution if $K = K^T$ satisfies the Riccati equation and $A_p - B_p(R + B_pKB_p^T)^{-1}B_p^TKA_p$ has all eigenvalues in the open unit disk. Note that when the (discrete or continuous) stabilizing solution exists it is unique. Moreover, if $Q = Q^T \geq 0$, the stabilizing solution is positive semidefinite.

2 The OCC Algorithm for Continuous Systems

2.1 The OCC Algorithm for State Feedback

In this section we consider the case of state feedback. With the state feedback controller (1.2) the closed loop system matrices in (1.4) are given by

$$A = A_p + B_pG \; ; \; D = D_p \; ; \; C_y = C_p \; ; \; C_u = G \; . \tag{2.1}$$

The following theorem provides conditions for optimality in the state feedback case.

Theorem 2.1 *Suppose there exists a matrix*

$$Q^* = block\,diag[Q_1^*, Q_2^*, \ldots, Q_m^*] \geq 0 \; ; \; Q_i^* = Q_i^{*T} \in R^{m_i \times m_i} \; ; \; i = 1, 2, \ldots, m. \tag{2.2}$$

such that the algebraic Riccati equation

$$0 = A_p^TK + KA_p - KB_pR^{-1}B_p^TK + C_p^TQ^*C_p \tag{2.3}$$

has the (unique) stabilizing solution K^*. Define

$$G^* = -R^{-1}B_p^TK^* \; , \tag{2.4}$$

and let X^* denote the unique solution of the Lyapunov equation

$$0 = (A_p + B_pG^*)X + X(A_p + B_pG^*)^T + D_pW_pD_p^T \; , \tag{2.5}$$

and define $Y_i = C_iX^*C_i^T$ $(i = 1, 2, \ldots, m)$. Then, if

$$0 = (Y_i - \overline{Y}_i)Q_i^* \; and \; Y_i \leq \overline{Y}_i \tag{2.6}$$

for all $i = 1, 2, \ldots, m$, we have that G^* given by (2.4) is an optimal solution to the OCC problem defined in (1.7).

Proof: Let Q^* be given by (2.2) and define the following LQ problem.

$$\min_{(G,X)} J(G, X) = trace\; RGXG^T + \sum_{i=1}^{m} trace\,(C_iXC_i^T - \overline{Y}_i)Q_i^* \tag{2.7}$$

subject to $A_p + B_pG$ stable and

$$0 = (A_p + B_pG)X + X(A_p + B_pG)^T + D_pW_pD_p^T \; . \tag{2.8}$$

Using a simple completion of square argument, it is easy to see, from (2.3), (2.4) and (2.5), that (G^*, X^*) solves (2.7).

Now let G denote a feasible controller (arbitrary but fixed) for the OCC problem. That is, $(A_p + B_p G)$ is stable and $C_i X C_i^T \leq \overline{Y}_i$ (for all $i = 1, 2, \ldots, m$), where X is the closed loop gramian corresponding to G. From the previous paragraph, we get that

$$\begin{aligned} Jocc(G^*, X^*) &= trace\ RG^* X^* (G^*)^T + \textstyle\sum_{i=1}^m trace\ (C_i X^* C_i^T - \overline{Y}_i) Q_i^* \\ &\leq trace\ RGXG^T + \textstyle\sum_{i=1}^m trace\ (C_i X C_i^T - \overline{Y}_i) Q_i^* \\ &\leq trace\ RGXG^T . \end{aligned} \tag{2.9}$$

Using the fact that $0 = (C_i X^* C_i^T - \overline{Y}_i) Q_i^*$, from (2.9), we obtain

$$trace\ RG^* X^* (G^*)^T \leq trace\ RGXG^T . \tag{2.10}$$

This last inequality together with the fact that G^* is also feasible for the OCC problem, because $C_i X^* C_i^T \leq \overline{Y}_i$ (for all $i = 1, 2, \ldots, m$), imply that G^* is a solution to the OCC problem. □

From (2.3) and (2.4), it follows that the solution of the OCC problem with static state feedback is an LQ controller with a special choice of output weighting matrix Q. Therefore, our algorithm for solving the conditions in Theorem 2.1 only needs to iterate on Q.

Before giving the algorithm we would like to mention that the existence of Q^* satisfying the conditions of Theorem 2.1 is necessary in certain cases. For example, from Theorem 5.8 of [5], it follows that, when the constraints in (1.6) are scalar, and (for example) the pairs $(C_1, A_p), \ldots, (C_m, A_p)$ do not have imaginary axis unobservable modes, then a diagonal Q^* exists if a solution to the OCC problem exists. See also reference [3]. The case of block diagonal matrices Q does not seem to appear in the published literature. It should be noted that the emphasis of the present paper is an algorithm for computing Q^* (and thus a controller that solves the OCC problem) under the assumption that a matrix Q^* satisfying the conditions of Theorem 2.1 exists. This algorithm is given next.

To give this algorithm we need to introduce the following operator. Let M denote a real symmetric matrix and suppose that

$$M = [U_1\ U_2]\ block\ diag\ [E_p,\ E_n]\ [U_1\ U_2]^T \tag{2.11}$$

is the (real) Schur decomposition of M, where E_p and E_n are diagonal matrices containing the strictly positive and nonpositive eigenvalues of M, in decreasing order, respectively; and $[U_1\ U_2]$ is an orthogonal matrix. Define

$$\mathcal{P}[M] = \begin{cases} 0 & ;\ if\ M \leq 0 \\ U_1 E_p U_1^T & ;\ otherwise . \end{cases} \tag{2.12}$$

Note that if M is a symmetric matrix with block diagonal structure, the operator $\mathcal{P}[\cdot]$ preserves the block structure, i.e., $\mathcal{P}[M]$ has the block structure of M.

The following algorithm for solving the conditions in Theorem 2.1 is the main contribution of this paper.

The OCC Algorithm

1) Given A_p, B_p, D_p, C_p, W_p, R, $\overline{Y}^b = block\ diag[\overline{Y}_1, \overline{Y}_2, \ldots, \overline{Y}_m]$, an initial point $Q(0) = block\ diag\ [Q_1(0), Q_2(0), \ldots, Q_m(0)] > 0$, and constants $\alpha > 0$, $0 < \beta < 1$, let $j = 0$ and go to 2).

2) Compute $K(j) \geq 0$ and $G(j)$ by solving

$$\begin{aligned} 0 &= A_p^T K(j) + K(j) A_p - K(j) B_p R^{-1} B_p^T K(j) + C_p^T Q(j) C_p \\ G(j) &= -R^{-1} B_p^T K(j) \ . \end{aligned} \tag{2.13}$$

3) Compute $X(j)$ by solving

$$0 = [A_p + B_p G(j)] X(j) + X(j) [A_p + B_p G(j)]^T + D_p W_p D_p^T. \tag{2.14}$$

4) Set $Y_i(j) = C_i X(j) C_i^T$ for $i = 1, 2, \ldots, m$.

5) Let $Y^b(j) = block\ diag\ [Y_1(Q(j)), Y_2(Q(j)), \ldots, Y_m(Q(j))]$, and

$$Q(j+1) = \beta Q(j) + (1-\beta)\mathcal{P}[Q(j) + \alpha\{Y^b(j) - \overline{Y}^b\}] \ , \tag{2.15}$$

$j = j + 1$, go to 2). □

Several stopping criteria may be used to guarantee that the OCC algorithm terminates in a finite number of steps. In this paper, we propose to stop the algorithm whenever the first equation in condition (2.6) is satisfied to a given numerical accuracy. This can be achieved by checking if the inequality

$$\sum_{i=1}^{m} ||(Y_i(j) - \overline{Y}_i) Q_i(j)|| < \epsilon \tag{2.16}$$

holds, where $\epsilon > 0$ is the specified tolerance. Inequality (2.16) must be tested after step 4). If (2.16) holds, we stop the algorithm, and declare $G(j)$, $Q_1(j)$, $Q_2(j)$, and $Q_m(j)$ to be a numerical solution to the OCC problem; if (2.16) does not hold, the algorithm continues.

The rest of this section is devoted to show that, under the assumption

(A1) (A_p, B_p) is stabilizable and A_p has no eigenvalues on the imaginary axis,

if there exists Q^* satisfying the conditions in Theorem 2.1 then the OCC algorithm will find it, provided α is properly chosen. More specifically, under the assumptions mentioned, we will show that the sequence of matrices $\{Q(j)\}_{j=0}^{\infty}$ generated by the OCC algorithm (see equation (2.15)) has a limit $\hat{Q}$ which satisfies all the conditions of Theorem 2.1. Thus, the OCC algorithm converges to a globally optimal solution to the OCC problem. Note that the existence of the limit $\hat{Q}$ implies that, given any $\epsilon > 0$, there exists an integer j such that inequality (2.16) holds.

Note that the OCC algorithm is well-posed in the sense that the unique positive semidefinite solution $K(j)$ to the Riccati equation (2.13) and the solution $X(j)$ to the Lyapunov equation (2.14) exist at each iteration. This follows from assumption (A1) and the fact that, at each iteration, $Q(j) \geq 0$. As is well known [7], since (A_p, B_p) is stabilizable and the pair

$(Q(j)^{1/2}C_p, A_p)$ has no imaginary axis unobservable modes, $K(j) \geq 0$ exists, it is unique, and it renders $A_p - B_p R^{-1} B_p^T K(j)$ asymptotically stable.

To establish our results we need to introduce the following operators

$$\begin{aligned} \mathcal{T}_\beta[Q] &:= \beta Q + (1-\beta)\mathcal{T}[Q]\,; \\ \mathcal{T}[Q] &:= \mathcal{P}[Q + \alpha\{Y^b(Q) - \overline{Y}^b\}]\,, \end{aligned} \tag{2.17}$$

where $\alpha > 0$ and $\beta \in (0,1)$ are parameters of the OCC algorithm. Note that, with this notation, the sequence of matrices $Q(j)$ generated by the OCC algorithm is $\{\mathcal{T}_\beta^j[Q(0)]\}_{j=0}^\infty$, where $Q(0) \geq 0$ is block diagonal.

Theorem 2.2 *Consider the OCC algorithm and suppose that $\alpha > 0$, $0 < \beta < 1$, and that assumption (A1) holds. Suppose that the algorithm converges, that is, the sequence $\{\mathcal{T}_\beta^j[Q(0)]\}_{j=0}^\infty$ converges to Q^*. Then $Q^* :=$ block diag $[Q_1^*, Q_2^*, \ldots, Q_m^*]$ satisfies the sufficient conditions in Theorem 2.1 for optimality. In other words, if the OCC algorithm converges, the resulting controller $u = -R^{-1}B_p^T K^* x$, where K^* solves (2.3), is a global optimal solution to the given OCC problem, where Q^* is the limit of the convergent sequence $\{\mathcal{T}_\beta^j[Q(0)]\}_{j=0}^\infty$.*

The proof of Theorem 2.2 requires the following lemma.

Lemma 2.1 *For any symmetric matrices $M = M^T$ and $N = N^T$ of the same dimensions, the following statements hold:*

1. *$\mathcal{P}[M+N] = M$ if and only if $M \geq 0$, $N \leq 0$ and $MN = 0$.*
2. *$\|\mathcal{P}[M] - \mathcal{P}[N]\| \leq \|M - N\|$, where $\|\cdot\|$ denotes the Frobenius norm.*

Proof: First, we shall show the necessity part of 1. The property $M \geq 0$ is a direct consequence of the definition of $\mathcal{P}[\cdot]$ in (2.12). Next, we show that $N \leq 0$ and $MN = 0$. Let $M + N$ have the Schur decomposition

$$\begin{aligned} M + N &= [U_1\ U_2]\mathit{block\ diag}[E_p\ ,\ E_n][U_1\ U_2]^T \\ &= U_1 E_p U_1^T + U_2 E_n U_2^T, \end{aligned} \tag{2.18}$$

where $E_p > 0$ and $E_n \leq 0$. Thus, from (2.12) and $\mathcal{P}[M+N] = M$ we obtain

$$\mathcal{P}[M+N] = U_1 E_p U_1^T = M\,. \tag{2.19}$$

Subtracting this last equation from (2.18) we obtain

$$N = U_2 E_n U_2^T \leq 0. \tag{2.20}$$

Since $U_1^T U_2 = 0$, from (2.19) and (2.20) it follows that

$$MN = U_1 E_p U_1^T U_2 E_n U_2^T = 0.$$

Second, we shall show the sufficiency part of 1. Let $M \geq 0$ and $N \leq 0$ be given and suppose that $MN = 0$. Note that if either M or N is zero, the sufficiency of property 1) is

trivial. Now suppose that $M \neq 0$ and $N \neq 0$. The real Schur decompositions of M and N are

$$M = U_1 E_p U_1^T \; ; \; N = U_2 E_n U_2^T ,$$

where $E_p > 0$, $E_n < 0$, $U_1^T U_1 = I$, $U_2^T U_2 = I$ and $U_1^T U_2 = 0$. Then,

$$\mathcal{P}[M+N] = \mathcal{P}[U_1 E_p U_1^T + U_2 E_n U_2^T] = U_1 E_p U_1^T = M . \tag{2.21}$$

Finally, we show property 2). Let M and N have the following Schur decompositions:

$$\begin{aligned} M &= [U_1 \; U_2] block\ diag[E_p\,,\; E_n][U_1 \; U_2]^T \; ; \; E_p > 0 \; ; \; E_n \leq 0; \\ N &= [\hat{U}_1 \; \hat{U}_2] block\ diag[\hat{E}_p\,,\; \hat{E}_n][\hat{U}_1 \; \hat{U}_2]^T \; ; \; \hat{E}_p > 0 \; ; \; \hat{E}_n \leq 0. \end{aligned} \tag{2.22}$$

Let

$$\begin{aligned} M^+ &:= \mathcal{P}[M] = U_1 E_p U_1^T > 0, \\ M^- &:= U_2 E_n U_2^T \leq 0, \\ N^+ &:= \mathcal{P}[N] = \hat{U}_1 \hat{E}_p \hat{U}_1^T > 0, \\ N^- &:= \hat{U}_2 \hat{E}_n \hat{U}_2^T \leq 0. \end{aligned}$$

Note that $M = M^+ + M^-$, $N = N^+ + N^-$, $M^+M^- = 0$, and $N^+N^- = 0$. Then,

$$\begin{aligned} \|M - N\|^2 &= \|(M^+ - N^+) + (M^- - N^-)\|^2 \\ &= \|M^+ - N^+\|^2 + \|M^- - N^-\|^2 \\ &\quad -2traceM^-N^+ - 2traceM^+N^- \end{aligned} \tag{2.23}$$

Since $-traceM^-N^+ \geq 0$ and $-traceM^+N^- \geq 0$, we obtain

$$\|M - N\|^2 \geq \|M^+ - N^+\|^2 = \|\mathcal{P}[M] - \mathcal{P}[N]\|^2 , \tag{2.24}$$

which completes the proof. □

The following lemma, essentially due to [2], is also required for the proof Theorem 2.2.

Lemma 2.2 *Consider the plant defined in (1.1) and suppose that assumption (A1) holds. Let K denote the unique stabilizing solution to the Riccati equation (2.3) with $Q = Q^T \geq 0$. Then $K(Q)$ is a real analytic function of $Q = Q^T \geq 0$.*

Proof of Theorem 2.2: By Lemma 2.2, the state feedback control gain $G(Q)$ in (2.4) is a continuous function of $Q = Q^T \geq 0$. Hence, the block output covariance matrix $Y^b(Q)$ is continuous with respect to $Q = Q^T \geq 0$. Since the operator $\mathcal{P}[\cdot]$ is continuous, we obtain that $\mathcal{T}[\cdot]$ and $\mathcal{T}_\beta[\cdot]$ are well defined and continuous for any $Q = Q^T \geq 0$. Suppose that $\{\mathcal{T}_\beta^j[Q(0)]\}_{j=0}^{\infty}$ converges to Q^*, i.e.,

$$\lim_{j \to \infty} \mathcal{T}_\beta^j[Q(0)] = Q^*. \tag{2.25}$$

Since $\beta \in (0,1)$ and $\mathcal{P}[\cdot]$ preserves the block structure, we may conclude that, for each j, $\mathcal{T}_\beta^j[Q(0)]$ has the correct block diagonal structure and it is positive semidefinite. Thus, $Q^* =$ *block diag* $[Q_1^*, Q_2^*, \ldots, Q_m^*]$ and $Q^* \geq 0$.

From the continuity properties of $\mathcal{T}_\beta$, we obtain

$$\mathcal{T}_\beta[Q^*] = \mathcal{T}_\beta\{\lim_{j\to\infty} \mathcal{T}_\beta^j[Q(0)]\} = \lim_{j\to\infty} \mathcal{T}_\beta^{j+1}[Q(0)] = Q^*. \quad (2.26)$$

That is, Q^* is a fixed point of $\mathcal{T}_\beta[\cdot]$. Since $\beta \neq 1$, from (2.17), we get

$$Q^* = \mathcal{T}[Q^*] = \mathcal{P}[Q^* + \alpha\{Y^b(Q^*) - \overline{Y}^b\}] . \quad (2.27)$$

Let $M = Q^*$ and $N = \alpha[Y^b(Q^*) - \overline{Y}^b]$. From Lemma 2.1, we conclude that

$$\alpha[Y^b(Q^*) - \overline{Y}^b] \leq 0 \;\; and \;\; \alpha Q^*[Y^b(Q^*) - \overline{Y}^b] = 0 .$$

Since $\alpha > 0$, the above inequalities imply that Q^* satisfies (2.6). Hence, Q^* satisfies the conditions in Theorem 2.1. This completes the proof. □

The following result shows that there is always a choice for the parameter α in the OCC algorithm that will guarantee its convergence, provided the conditions in Theorem 2.1 admit one solution.

Theorem 2.3 *Suppose that assumption (A1) holds. Assume also that there exists Q^* satisfying the conditions in Theorem 2.1. Then, given any $Q(0) \geq 0 \in \mathcal{R}^{n_y \times n_y}$ with the appropriate block diagonal structure, there exists an $\alpha^* > 0$ such that if $0 < \alpha \leq \alpha^*$, the sequence $\{\mathcal{T}_\beta^n[Q(0)]\}_{n=0}^\infty$ will converge to some $\hat{Q} \geq 0$ satisfying the conditions in Theorem 2.1. That is, the OCC algorithm will converge to a global optimal solution of the given OCC problem.*

In order to prove Theorem 2.3, we need a few intermediate results and definitions. Let $Q = Q^T \geq 0$ be given and let K denote the (unique) stabilizing solution to

$$0 = A_p^T K + K A_p - K B_p R^{-1} B_p^T K + C_p^T Q C_p . \quad (2.28)$$

Then, with the state feedback gain $G = -R^{-1} B_p^T K$, the l-th output covariance of the closed loop Y_l $(l = 1, 2, \cdots, m)$ is given by

$$Y_l = C_l X C_l^T ,$$

where X is the unique solution to

$$0 = (A_p + B_p G)X + X(A_p + B_p G)^T + D_p W_p D_p^T . \quad (2.29)$$

Now, let

$$Q = block\ diag[Q_1, Q_2, \ldots, Q_m] \; ; \; Q_l := [q_{ij}^l] \in \mathcal{R}^{m_l \times m_l} , \quad (2.30)$$

and

$$Y^b = block\ diag[Y_1, Y_2, \cdots, Y_m] . \quad (2.31)$$

Below, we compute the derivative of Y^b with respect to the weighting matrix Q given in (2.30). We do this using vector notation. Let Q be given by (2.30) and define the operator $svec$ by

$$svec[Q] = \begin{bmatrix} q^1 \\ q^2 \\ \vdots \\ q^m \end{bmatrix} \in \mathcal{R}^n , \quad (2.32)$$

where

$$q^i := \sqrt{2}\left[\frac{q^i_{11}}{\sqrt{2}}, q^i_{12}, \ldots, q^i_{1m_i}, \frac{q^i_{22}}{\sqrt{2}}, q^i_{23}, \ldots, q^i_{2m_i}, \ldots, \frac{q^i_{m_i m_i}}{\sqrt{2}}\right]^T . \tag{2.33}$$

Note also that the operator *svec* defined in (2.32) preserves the Frobenius norm; i.e., if Q is given by (2.30) we have

$$||Q|| = ||svec[Q]|| . \tag{2.34}$$

Moreover, $sevc[\cdot]$ is a linear operator.

Let

$$y = svec[Y^b] , \tag{2.35}$$

where Y^b is given by (2.31). Define also the following symmetric matrix

$$E_i = svec^{-1}[e_i] , \tag{2.36}$$

where $e_i \in \mathcal{R}^n$ has a one in the i-th row and zeros elsewhere, and $svec^{-1}$ is the inverse of the operator *svec*.

Lemma 2.3 *Consider the system defined in (1.1) and suppose assumption (A1) holds. Let* $Q = Q^T \geq 0$ *be given by (2.30) and define* $q = svec[Q]$. *Let* y *be given by (2.35). Then, the partial derivative of* $y \in \mathcal{R}^n$ *with respect to* $q \in \mathcal{R}^n$ *is*

$$\frac{\partial y}{\partial q} = -[H_{ij}] = -[2trace(P_i B_p R^{-1} B_p^T P_j X)] \; ; \; i, j = 1, 2, \ldots, n, \tag{2.37}$$

where P_i *is the unique solution to*

$$0 = P_i(A_p + B_p G) + (A_p + B_p G)^T P_i + C_p^T E_i C_p \tag{2.38}$$

with E_i *given by (2.36). Moreover, if* $Q = Q^T \geq 0$, *the matrix-valued function* $H(Q) = [H_{ij}]$ *is continuous and it satisfies* $H(Q) \geq 0$.

Proof: Let y_i denote the i-th component of y. From the definition of the operator *svec* (see, for example, (2.32)) it follows that

$$y_i = trace(E_i C_p X C_p^T) . \tag{2.39}$$

Using the Lyapunov equations (2.29) and (2.38) it follows from (2.39) that

$$y_i = trace(P_i D_p W_p D_p^T) , \tag{2.40}$$

where P_i is the solution to (2.38). Hence, from (2.40), we get

$$\frac{\partial y_i}{\partial q_j} = trace(P_{ij} D_p W_p D_p^T), \tag{2.41}$$

where q_j is the j-th component of q and $P_{ij} = \frac{\partial P_i}{\partial q_j}$.

Now to generate P_{ij}, differentiate equation (2.38) with respect to q_j to obtain

$$\begin{aligned} 0 &= P_{ij}(A_p + B_pG) + (A_p + B_pG)^T P_{ij} \\ &- P_iB_pR^{-1}B_p^T\frac{\partial K}{\partial q_j} - \frac{\partial K}{\partial q_j}B_pR^{-1}B_p^TP_i, \end{aligned} \tag{2.42}$$

From the Riccati equation (2.28) and the Lyapunov equation (2.38) we get

$$P_j = \frac{\partial K}{\partial q_j} ,$$

where P_j solves (2.38) with the "E-matrix" equal to E_j. Hence, from (2.42), we obtain

$$\begin{aligned} P_{ij} &= -\int_0^\infty \exp[(A_p + B_pG)^T t][P_iB_pR^{-1}B_p^TP_j \\ &\quad +P_jB_pR^{-1}B_p^TP_i]\exp[(A_p + B_pG)t]dt. \end{aligned} \tag{2.43}$$

Finally, from (2.29), (2.41), and (2.43) we obtain

$$\frac{\partial y}{\partial q} = -[H_{ij}] = -[2trace(P_iB_pR^{-1}B_p^TP_jX)] , \tag{2.44}$$

which gives (2.37).

The continuity of $H(Q)$ follows from the fact that, on the set of positive semidefinite matrices Q, the matrix-valued functions P_i, P_j, and X are all continuous. Note also that

$$\begin{aligned} H_{ij} &= trace\ (P_iB_pR^{-1}B_p^TP_jX) \\ &= < X^{1/2}P_iB_pR^{-1/2}, X^{1/2}P_jB_pR^{-1/2} > , \end{aligned}$$

where $< M, N >= traceMN^T$ is the standard inner product on the space of matrices $\mathcal{R}^{n_x \times n_u}$, where n_x and n_u are dimensions of the plant states and controls. Thus, H is of the form

$$H_{ij} = \left[< X^{1/2}P_iB_pR^{-1/2}, X^{1/2}P_jB_pR^{-1/2} >\right] , \tag{2.45}$$

which shows that $H \geq 0$. □

The following results may be found in [11], see Proposition 3.2.3 and Proposition 12.3.7.

Lemma 2.4 *Assume that $\mathcal{F} : \mathcal{R}^n \rightarrow \mathcal{R}^n$ is Frechet-differentiable on a convex set $\mathcal{D}'_0 \subset \mathcal{R}^n$. Then for any x and $y \in \mathcal{D}'_0$,*

$$\|\mathcal{F}(y) - \mathcal{F}(x)\| \leq \sup_{0 \leq t \leq 1} \overline{\sigma}\{\mathcal{F}'[x + t(y - x)]\}\|x - y\| , \tag{2.46}$$

where $\mathcal{F}'(\cdot)$ denotes the Frechet-derivative of $\mathcal{F}(\cdot)$, and $\overline{\sigma}[\cdot]$ denotes the maximum singular value of $[\cdot]$.

Lemma 2.5 *Suppose that $\mathcal{T} : \mathcal{R}^{n_y \times n_y} \rightarrow \mathcal{R}^{n_y \times n_y}$ is nonexpansive on the closed, convex set $\mathcal{D}_0$. That is, for any $x, y \in \mathcal{D}_0$, we have*

$$\|\mathcal{T}(y) - \mathcal{T}(x)\| \leq \|y - x\| . \tag{2.47}$$

Assume, further, that $\mathcal{T}\mathcal{D}_0 \subset \mathcal{D}_0$ and that $\mathcal{D}_0$ contains a fixed point of $\mathcal{T}$. Then for any $\beta \in (0,1)$ and $x^0 \in \mathcal{D}_0$ the iteration

$$x^{k+1} = \beta x^k + (1-\beta)\mathcal{T}(x^k) \; ; \; k = 0, 1, \ldots , \tag{2.48}$$

converges to a fixed point of $\mathcal{T}$ in $\mathcal{D}_0$.

Proof of Theorem 2.3: The proof of Theorem 2.3 consists of two steps. First, we show the nonexpansive property of operator $\mathcal{T}$ defined in (2.17). By assumption, there exists Q^* satisfying the conditions in Theorem 2.1. Define a subset of $\mathcal{R}^{n_y \times n_y}$ as follows

$$\begin{aligned} \mathcal{D}_0 \;&:=\; \{Q \geq 0 \in \mathcal{R}^{n_y \times n_y} : Q = block\ diag[Q_1, Q_2, \ldots, Q_m] \\ &\quad\; and \, \|Q - Q^*\| \leq \|Q(0) - Q^*\|\} \,. \end{aligned} \tag{2.49}$$

where n_y is the dimension of y_p, and $Q(0)$ is the initial output weighting matrix for the OCC algorithm. It is obvious that the set $\mathcal{D}_0$ is compact (i.e., closed and bounded) and convex. Let $\mathcal{D'}_0$ be a set defined by

$$\mathcal{D'}_0 := \{q = svec[Q] \in \mathcal{R}^n : Q \in \mathcal{D}_0\} \,. \tag{2.50}$$

It is clear that $\mathcal{D'}_0$ is convex, for $svec[\cdot]$ is a linear operator and $\mathcal{D}_0$ is convex. Let $q \in \mathcal{D'}_0$ and define $y(q) = svec[Y^b(Q)]$, and

$$\mathcal{F}[q] = q + \alpha y(q) \,. \tag{2.51}$$

Note that $\mathcal{F}[\cdot]$ is well-defined and Frechet-differentiable, with respect to q, in $\mathcal{D'}_0$. In fact, from Lemma 2.3, it follows that the Frechet derivative of $\mathcal{F}[\cdot]$ is

$$\mathcal{F}'[q] = I - \alpha H(q) \,, \tag{2.52}$$

where $H(q)$ is defined in (2.37). (Here, we think of H as a function of $q = svec[Q]$ instead of a function Q.)

Now, let Q^ν and Q^μ in $\mathcal{D}_0$ be given. Define $q^\nu = svec[Q^\nu]$ and $q^\mu = svec[Q^\mu]$. Then, since $svec[\cdot]$ preserves the Frobenius norm, we have

$$\begin{aligned} \|\mathcal{T}[Q^\nu] - \mathcal{T}[Q^\mu]\| \;&=\; \|\mathcal{P}[Q^\nu - \alpha\{Y^b(Q^\nu) - \overline{Y}^b\}] - \mathcal{P}[Q^\mu - \alpha\{Y^b(Q^\mu) - \overline{Y}^b\}]\| \\ &\leq\; \|Q^\nu - Q^\mu - \alpha[Y^b(Q^\nu) - Y^b(Q^\mu)]\| \\ &=\; \|q^\nu - q^\mu - \alpha[y^\nu - y^\mu]\|, \end{aligned} \tag{2.53}$$

where $y^\nu := svec[Y^b(Q^\nu)]$ and $y^\mu := svec[Y^b(Q^\mu)]$. Since q^ν and q^μ belong to $\mathcal{D'}_0$, using Lemma 2.4, we have

$$\begin{aligned} \|q^\nu - q^\mu - \alpha[y^\nu - y^\mu]\| \;&=\; \|\mathcal{F}[q^\nu] - \mathcal{F}[q^\mu]\| \\ &\leq\; \sup_{0 \leq t \leq 1} \overline{\sigma}[\mathcal{F}'[tq^\nu + (1-t)q^\mu]]\|q^\nu - q^\mu\| \\ &=\; \sup_{0 \leq t \leq 1} \overline{\sigma}[I - \alpha H\{[tq^\nu + (1-t)q^\mu]\}]\|q^\nu - q^\mu\| \end{aligned} \tag{2.54}$$

Since H is a continuous function over the compact set $\mathcal{D}'_0$, there exists an $\alpha^* > 0$ such that for any $q^\nu \in \mathcal{D}'_0$, $q^\mu \in \mathcal{D}'_0$, and $0 \leq t \leq 1$, we have

$$\bar{\sigma}\{H[tq^\nu + (1-t)q^\mu]\} \leq 2/\alpha^*. \tag{2.55}$$

Thus, since for any q^ν and $q^\mu \in \mathcal{D}'_0$, and any $t \in [0,1]$, $H[tq^\nu + (1-t)q^\mu] \geq 0$, we have

$$\sup_{0\leq t\leq 1} \bar{\sigma}\{I - \alpha H[tq^\nu + (1-t)q^\mu]\} \leq 1 \tag{2.56}$$

for any $\alpha \leq \alpha^*$. Therefore, using (2.53) and (2.54), for any $\alpha \leq \alpha^*$ we obtain

$$\|\mathcal{T}[Q^\nu] - \mathcal{T}[Q^\mu]\| \leq \|q^\nu - q^\mu\| = \|Q^\nu - Q^\mu\|. \tag{2.57}$$

Hence, for any $\alpha \leq \alpha^*$, the operator $\mathcal{T}$ is nonexpansive on $\mathcal{D}_0$. Replacing Q^μ by Q^* proves that for any $Q \in \mathcal{D}_0$

$$\|\mathcal{T}[Q] - \mathcal{T}[Q^*]\| \leq \|Q - Q^*\| \leq \|Q(0) - Q^*\|. \tag{2.58}$$

Now, using Lemma 2.1 and the fact that Q^* satisfies the conditions of Theorem 2.1, we conclude that $\mathcal{T}[Q^*] = Q^*$. This equation and (2.58) imply

$$\|\mathcal{T}[Q] - Q^*\| \leq \|Q(0) - Q^*\|, \tag{2.59}$$

therefore, $\mathcal{T}[Q] \in \mathcal{D}_0$.

Second, we shall show the convergence of the OCC algorithm, that is, the convergence of the sequence $\{\mathcal{T}^j_\beta[Q(0)]\}_{j=0}^\infty$. Since $\mathcal{T}\mathcal{D}_0 \subset \mathcal{D}_0$, $\mathcal{D}_0$ is convex, and it contains a fixed point of $\mathcal{T}$, from Lemma 2.5, we obtain that the sequence $\{\mathcal{T}^j_\beta[Q(0)]\}_{j=0}^\infty$ generated by the iteration

$$Q(j+1) = \mathcal{T}_\beta[Q(j)] = \beta Q(j) + (1-\beta)\mathcal{T}[Q(j)] \tag{2.60}$$

converges to a fixed point of $\mathcal{T}$ in $\mathcal{D}_0$, say $\hat{Q}$. The fact that $\hat{Q}$ satisfies the sufficient conditions in Theorem 2.1 is the direct consequence of Theorem 2.2. □

2.2 The OCC Algorithm for Full Order Dynamic Feedback

The extension of the state feedback case to the full order dynamic feedback case is straight forward. In fact, the state feedback OCC algorithm can be applied to solve the full order dynamic feedback OCC problem. Here, for system (1.1), we assume that assumption (A1) holds and that

(A2) (M_p, A_p) is detectable.

As is well known [7], under assumption (A2), there exists a unique matrix $\bar{X}$ that satisfies the Riccati equation

$$0 = A_p\bar{X} + \bar{X}A_p^T - \bar{X}M_p^TV^{-1}M_p\bar{X} + D_pW_pD_p^T \tag{2.61}$$

and $A_p - \bar{X}M_p^TV^{-1}$ is asymptotically stable. Moreover, $\bar{X} \geq 0$. With this matrix $\bar{X}$, we define

$$F = \bar{X}M_p^TV^{-1}\,. \tag{2.62}$$

Theorem 2.4 *Consider the plant defined in (1.1). Let $\tilde{X}$ and F denote the matrices in (2.61) and (2.62). Suppose there exists a matrix*

$$Q^* = block\ diag[Q_1^*, Q_2^*, \ldots, Q_m^*] \geq 0\ ;\ Q_i^* = Q_i^{*T} \in \mathcal{R}^{m_i \times m_i}\ ;\ i = 1, 2, \ldots, m, \tag{2.63}$$

such that the algebraic Riccati equation

$$0 = A_p^T K + K A_p - K B_p R^{-1} B_p^T K + C_p^T Q^* C_p \tag{2.64}$$

has the (unique) stabilizing solution K^*. Define

$$G = -R^{-1} B_p^T K^*\ , \tag{2.65}$$

and let X^* denote the unique solution to the Lyapunov equation

$$0 = (A_p + B_p G) X + X (A_p + B_p G)^T + F V F^T\ , \tag{2.66}$$

and define $Y_i = C_i(\tilde{X} + X^*)C_i^T$ $(i = 1, 2, \ldots, m)$. Then, if

$$0 = (Y_i - \overline{Y}_i) Q_i^* \text{ and } Y_i \leq \overline{Y}_i \tag{2.67}$$

for all $i = 1, 2, \ldots, m$, the dynamic controller

$$\begin{aligned} \dot{x}_c(t) &= (A_p + B_p G - F M_p) x_c(t) + F z(t) \\ u(t) &= G x_c(t) \end{aligned} \tag{2.68}$$

is an optimal solution to the OCC problem defined in (1.7).

A proof of this theorem may be obtained by combining Theorem 2.1 in this paper, and Lemma 4.2 and Theorem 4.1 in [12]. The result in [12] shows how to reduce the OCC problem (and other $\mathcal{H}_2$-like problems) with output feedback to an equivalent problem with state feedback.

Note that the matrices $\tilde{X}$ and F in (2.61) and (2.62) do not depend on the weighting matrix Q^*. To find a matrix Q^* satisfying the conditions in Theorem 2.4, we can use the OCC algorithm given in Section 2. This requires that, in the OCC algorithm, we replace D_p, W_p, and $\overline{Y}_i$ by F, V, and $\overline{Y}_i - C_i \tilde{X} C_i^T$, respectively.

3 Discrete-Time Version

The discrete-time version of the OCC problem is very much like the continuous-time case. Here, we give the definition of the OCC problem and the main results.

Consider the following discrete system

$$\begin{aligned} x_p(k+1) &= A_p x_p(k) + B_p u(k) + D_p w_p(k) \\ y_p(k) &= C_p x_p(k) \\ z(k) &= M_p x_p(k) + v(k)\ . \end{aligned} \tag{3.1}$$

Suppose that to the plant (3.1) we apply a full state feedback stabilizing control, i.e.,

$$u(k) = Gx(k) \tag{3.2}$$

or a strictly proper stabilizing control

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + Fz(k) \\ u(k) &= Gx_c(k)\ . \end{aligned} \tag{3.3}$$

Then the closed loop system has the following form

$$\begin{aligned} x(k+1) &= Ax(k) + Dw(k) \\ y(k) &= \begin{bmatrix} y_p(k) \\ u(k) \end{bmatrix} = \begin{bmatrix} C_y \\ C_u \end{bmatrix} x(k) = Cx(k)\ , \end{aligned} \tag{3.4}$$

where the definitions of matrices A, B, C and vectors x, w, y are as in the continuous-time case.

As in Section 1, let $W_p > 0$ and $V > 0$ denote symmetric matrices with dimensions equal to the dimensions of w_p and z, respectively. Define $W = W_p$ if (3.2) is used in (3.4), or $W = \mathit{block\ diag}[W_p,\ V]$ if (3.3) is used. Let X denote the closed loop controllability gramian from the input $W^{-1/2}w$. Since A is stable, X is given by

$$X = AXA^T + DWD^T\ . \tag{3.5}$$

As in the continuous-time case, we seek a solution to the following optimal control problem:

<u>The Discrete-Time OCC Problem</u>

Find a state feedback stabilizing controller (3.2) or a strictly proper output feedback stabilizing controller (3.3) for the system (3.1) to minimize the OCC cost

$$J_{OCC} = trace R C_u X C_u^T\ ,\ R > 0 \tag{3.6}$$

subject to

$$Y_i = C_i X C_i^T \leq \overline{Y}_i\ ;\ i = 1, 2, \ldots, m, \tag{3.7}$$

where X is given by (3.5). □

The discrete-time OCC problem has interpretations similar to the ones of the continuous-time case. For example, the discrete-time OCC problem may be interpreted as the problem of minimizing a weighted sum of the worst-case peak values of the control signals u_i subject to constraints on the worst case peak values of the response y_i, when the disturbance w is unknown but has bounded energy. This is because, as in the continuous-time case, discrete-time gains from ℓ_2 to ℓ_∞ may also be computed using controllability gramians [18].

3.1 State Feedback Case

In this sub-section we consider the case of state feedback. The following theorem provides conditions for global optimality. Its proof is similar to that of Theorem 2.1 and it is omitted.

Theorem 3.1 *Suppose there exists a matrix*

$$Q^* = block\,diag[Q_1^*, Q_2^*, \ldots, Q_m^*] \geq 0\ ;\ Q_i^* = Q_i^{*T} \in R^{m_i \times m_i}\ ;\ i = 1, 2, \ldots, m. \tag{3.8}$$

such that the algebraic Riccati equation

$$K = A_p^T K A_p - A_p^T K B_p (R + B_p^T K B_p)^{-1} B_p^T K A_p + C_p^T Q^* C_p \tag{3.9}$$

has the (unique) stabilizing solution K^*. Define

$$G^* = -(R + B_p^T K^* B_p)^{-1} B_p^T K^* A_p\ , \tag{3.10}$$

and let X^* denote the unique solution of the Lyapunov equation

$$X = (A_p + B_p G^*) X (A_p + B_p G^*)^T + D_p W_p D_p^T\ , \tag{3.11}$$

and define $Y_i = C_i X^* C_i^T$ $(i = 1, 2, \ldots, m)$. Then, if

$$0 = (Y_i - \overline{Y}_i) Q_i^* \text{ and } Y_i \leq \overline{Y}_i \tag{3.12}$$

for all $i = 1, 2, \ldots, m$, we have that G^* given by (3.10) is an optimal solution to the OCC problem defined in (3.6).

The following algorithm may be used to find a matrix Q^*, and consequently a matrix G^* for the OCC problem, satisfying the conditions in Theorem 3.1.

The Discrete-Time OCC Algorithm

1) Given A_p, B_p, D_p, C_p, W_p, R, $\overline{Y}_i = block\,diag[\overline{Y}_1, \overline{Y}_2, \ldots, \overline{Y}_m]$, an initial point $Q(0) = block\,diag[Q_1(0), Q_2(0), \ldots, Q_m(0)] > 0$, and constants $\alpha > 0$, $0 < \beta < 1$, let $j = 0$ and go to 2).

2) Compute $K(j) \geq 0$ and $G(j)$ by solving

$$\begin{aligned} K(j) &= A_p^T K(j) A_p + C_p^T Q(j) C_p \\ &\quad - A_p^T K(j) B_p [R + B_p^T K(j) B_p]^{-1} B_p^T K(j) A_p \\ G(j) &= -[R + B_p^T K(j) B_p]^{-1} B_p^T K(j) A_p\ . \end{aligned} \tag{3.13}$$

3) Compute $X(j)$ by solving

$$X(j) = [A_p + B_p G(j)] X(j) [A_p + B_p G(j)]^T + D_p W_p D_p^T. \tag{3.14}$$

4) Set $Y_i(j) = C_i X(j) C_i^T$ for $i = 1, 2, \ldots, m$.

5) Let $Y^b(j) = block\ diag[Y_1(Q(j)), Y_2(Q(j)), \ldots, Y_m(Q(j))]$, and

$$Q(j+1) = \beta Q(j) + (1-\beta)\mathcal{P}[Q(j) + \alpha\{Y^b(j) - \overline{Y}^b\}], \qquad (3.15)$$

$j = j+1$, go to 2).

In (3.15), the operator $\mathcal{P}[\cdot]$ is as defined in (2.12). □

The same stop criterion as the continuous-time case is proposed for the discrete-time OCC algorithm. That is, equation (2.16) needs to be tested after step 4) for a given tolerance $\epsilon > 0$. If (2.16) holds, $G(j)$, $Q_1(j)$, $Q_2(j)$, and $Q_m(j)$ are numerical solutions for the given OCC problem; else, the algorithm continues.

In the rest of this section, we will assume the following:

(A3) (A_p, B_p) is stabilizable and A_p has no eigenvalues on the unit circle.

Note that the discrete-time OCC algorithm is well-posed in the sense that the unique positive semidefinite solution $K(j)$ to the Riccati equation (3.13) and the solution $X(j)$ to the Lyapunov equation (3.14) exist at each iteration. This follows from assumption (A3) and the fact that, at each iteration, $Q(j) \geq 0$. As is well known [8], since (A_p, B_p) is stabilizable and the pair $(Q(j)^{1/2}C_p, A_p)$ has no unobservable modes on the unit circle, $K(j) \geq 0$ exists, it is unique, and it renders $A_p - B_p(R + B_p^T K(j) B_p)^{-1} B_p^T K(j) A_p$ asymptotically stable.

A close examination of the proofs of the continuous-time results given in Theorems 2.2 and 2.3 reveals that the convergence property of the continuous-time algorithm follows from:

i) The properties of the operator $\mathcal{P}[\cdot]$ given in Lemma 2.1.

ii) The properties of the stabilizing solution to the continuous-time Riccati equation given in Lemma 2.2.

iii) The formula for the derivatives of the output covariance matrices $Y_1(Q)$, $Y_2(Q)$, ..., $Y_m(Q)$, with respect to Q, given in Lemma 2.3.

Certainly, property i) above holds in the discrete-time case because the operator $\mathcal{P}[\cdot]$ is the same. Also, it is relatively easy to show that, under assumption (A3), property ii) extends to the discrete-time setting. Finally, property iii) above also holds in the discrete-time case, provided that the Lyapunov equation (2.38) is replaced by its discrete-time counterpart, and the matrices R and X in (2.37) are replaced by $R + B_p^T K B_p$ and $X - D_p W_p D_p^T$, respectively. Thus, we may now conclude the following result.

Theorem 3.2 *Suppose that the assumption (A3) holds. Assume also that there exists Q^* satisfying the conditions in Theorem 3.1. Then, given any $Q(0) \geq 0 \in \mathcal{R}^{n_y \times n_y}$ with the appropriate block diagonal structure, there exists an $\alpha^* > 0$ such that if $0 < \alpha \leq \alpha^*$, the sequence $\{\mathcal{T}_\beta^j[Q(0)]\}_{j=0}^{\infty}$ will converge to some $\hat{Q} \geq 0$ satisfying the conditions in Theorem 3.1. That is, the discrete-time OCC algorithm will converge to a global optimal solution of the given OCC problem.*

3.2 Full Order Dynamic Feedback

As in the continuous-time case, the discrete-time state feedback results can be readily extended to solve the discrete-time OCC problem with output feedback.

Consider the system (3.1) and suppose that

(A4) (M_p, A_p) is detectable.

Then, there exists a unique matrix $\tilde{X}$ that satisfies the Riccati equation

$$\tilde{X} = A_p\tilde{X}A_p^T - A_p\tilde{X}M_p^T(V + M_p\tilde{X}M_p^T)^{-1}M_p\tilde{X}A_p^T + D_pW_pD_p^T \tag{3.16}$$

and $A_p - A_p\tilde{X}M_p^T(V + M_p\tilde{X}M_p^T)^{-1}M_p$ is asymptotically stable; see, for example, [8]. Moreover, $\tilde{X} \geq 0$. With this matrix $\tilde{X}$, we define

$$F = A_p\tilde{X}M_p^T(V + M_p\tilde{X}M_p^T)^{-1} . \tag{3.17}$$

The next result gives a solution to the OCC problem with strictly proper output feedback controllers, the proof follows the continuous-time case and it is omitted.

Theorem 3.3 *Consider the plant defined in (3.1). Let $\tilde{X}$ and F denote the matrices in (3.16) and (3.17). Suppose there exists a matrix*

$$Q^* = block\ diag[Q_1^*, Q_2^*, \ldots, Q_m^*] \geq 0 \ ; \ Q_i^* = Q_i^{*T} \in \mathcal{R}^{m_i \times m_i} \ ; \ i = 1, 2, \ldots, m, \tag{3.18}$$

such that the algebraic Riccati equation

$$K = A_p^TKA_p - A_p^TKB_p(R + B_p^TKB_p)^{-1}B_p^TKA_p + C_p^TQ^*C_p \tag{3.19}$$

has the (unique) stabilizing solution K^*. Define

$$G = -(R + B_p^TK^*B_p)^{-1}B_p^TK^*A_p \ , \tag{3.20}$$

and let X^* denote the unique solution to the Lyapunov equation

$$X = (A_p + B_pG)X(A_p + B_pG)^T + F(V + M_p\tilde{X}M_p^T)F^T \ , \tag{3.21}$$

and define $Y_i = C_i(\tilde{X} + X^*)C_i^T$ $(i = 1, 2, \ldots, m)$. Then, if

$$0 = (Y_i - \overline{Y}_i)Q_i^* \text{ and } Y_i \leq \overline{Y}_i \tag{3.22}$$

for all $i = 1, 2, \ldots, m$, the dynamic controller

$$\begin{aligned} x_c(k+1) &= (A_p + B_pG - FM_p)x_c(k) + Fz(k) \\ u(k) &= Gx_c(k) \end{aligned} \tag{3.23}$$

is an optimal solution to the OCC problem defined in (3.6).

Note that, as in the continuous-time case, the computation of $\tilde{X}$ and F are independent of the selection of the output weighting matrix Q. Hence, we can apply the discrete-time OCC algorithm with state feedback to solve the discrete-time full order output feedback OCC problem under the assumption that the optimal solutions are strictly proper. This requires that in the algorithm given in Section 3.1, we replace D_p, W_p, and $\overline{Y}_i$ by F, $V + M_p\tilde{X}M_p^T$, and $\overline{Y}_i - C_i\tilde{X}C_i^T$, respectively.

4 An Example

We consider the continuous-time OCC problem defined in (1.7) for the plant (1.1) with the following system matrices:

$$A_p = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -0.1 & 1 \\ 0 & 0 & -10 \end{bmatrix} ; \; B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ; \; D_p = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ; \tag{4.1a}$$

$$M_p = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} ; \tag{4.1b}$$

$$C_p = \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 1 & 1 & 0 \end{bmatrix} . \tag{4.1c}$$

Both the process noise w_p and the measurement noise v are scalar variables, while the performance variable y_p has three components. The weighting matrices required to define the OCC problem (1.7) are taken to be

$$W_p = 1 \; , \; V = 0.01 \; , and \; R = 1 \; . \tag{4.2}$$

Below, we consider two different OCC problems. These two problems differ in the grouping of the performance variables y_i used to define the constraints (1.6). For each problem, we consider both state-feedback and dynamic output feedback.

4.1 Problem 1

Here, the OCC problem has the performance constraints

$$Y_1 \leq 0.035 \;\; , \;\; Y_2 \leq 0.050 \;\; , \;\; Y_3 \leq 0.050 \;\; , \tag{4.3}$$

where Y_1, Y_2, and Y_3 denote the output covariance (1×1) matrices introduced in (1.6), corresponding to the first, second, and third performance variable respectively. Note that this OCC problem can be also solved by the ellipsoid algorithms given in [1, 3, 12] or the quadratically convergent algorithms given in [16].

First, we consider the case of state feedback. We use the algorithm described in Section 2.1 with the following parameters

$$Q(0) = I_3 \;\; , \;\; \beta = 0.1 \;\; , \;\; \epsilon = 10^{-6} \;\; . \tag{4.4}$$

To asses the effect of the user-specified parameter α, we ran the the algorithm with $1.0 \leq \alpha \leq 7.25$. Figure 1 shows the number of iterations required to meet the stopping criteria of the algorithm versus α. Clearly, as α approaches 1 or 7.25 the iteration number increases. From Figure 1, it follows that there exists an optimal α which uses the least number of iterations. Finding such an optimal α in terms of the system and specification matrices remains an open problem.

Table 1 shows the results of running the algorithm with $\alpha = 4.5$. Both state and output feedback cases are computed. In the state feedback case, G denotes the state feedback gain.

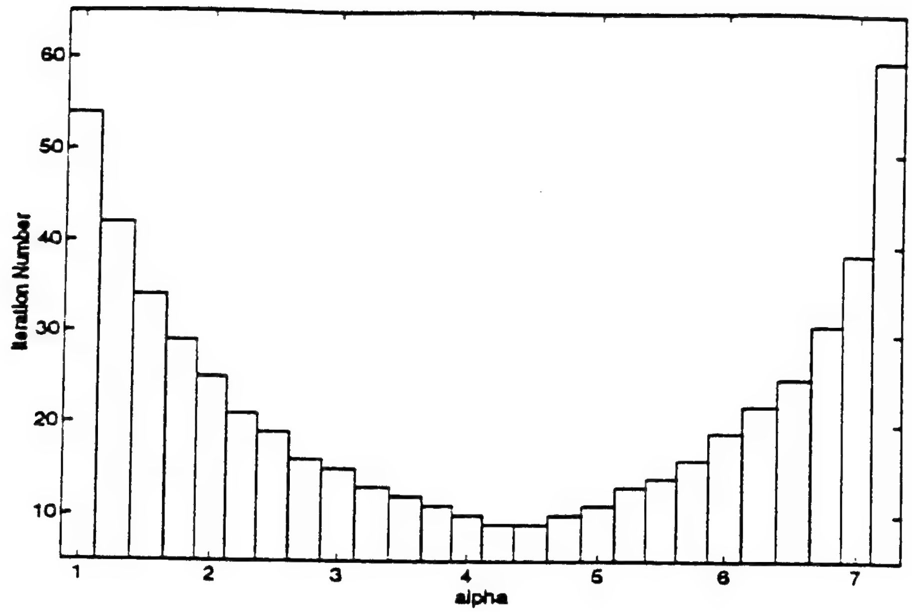

Figure 1: Iteration Number verses α

In the output feedback case, G denotes the controller output matrix, see (2.65) and (2.68). The controller input matrix F is precomputed according to (2.62). In this case we have

$$F = [\, 0.4412 \;\; 0.7633 \;\; 0.4796 \,]^T \; . \tag{4.5}$$

From Table 1, we can see that both controllers are feasible, for Y_i satisfies the bound $Y_i \leq \overline{Y}_i$. The only active constraint is the third one, i.e. $Y_3 = \overline{Y}_3$; hence, the corresponding output weight Q_3 is nonzero. As expected, the optimal cost J_{OCC} with output feedback is bigger than that with state feedback.

4.2 Problem 2

Now, the OCC problem has the performance constraints

$$Y_1 \leq 0.035 \;\; ; \;\; Y_2 \leq 0.050 \times I_2 \;\; , \tag{4.6}$$

where Y_1 denotes the (1×1) output covariance matrix corresponding to the first performance output, and Y_2 the (2×2) output covariance matrix of the second and third performance outputs grouped together.

Table 2 shows the results of running the algorithm with $\alpha = 30$ for both state and output feedback cases. The other parameters required by the algorithm are those in (4.4). For the output feedback case the input gain matrix F of the controller given in (4.5).

From Table 2, we can see that both controllers are feasible. As expected, the optimal cost J_{OCC} with output feedback is bigger than that with state feedback. Also, note that the constraint on the second output group $Y_2 \leq 0.05 \times I_2$ is sufficient for the output covariance constraints of Problem 1 in (4.3), that is, $Y_2 \leq 0.05$ and $Y_3 \leq 0.05$. As expected, the cost of Problem 2 for both state and output feedback cases are bigger than those of Problem 1, respectively.

State Feedback Design					
Iteration Number	Constraints		Optimal Cost		
	Spec. ($\overline{Y}_i$)	Actual (Y_i)	J_{OCC}	Q_i	G^T
9	0.0350 0.0500 0.0500	0.0314 0.0123 0.0500	0.0234	0.0000 0.0000 1.4268	0.0237 -0.9522 -0.0948
Output Feedback Design					
Iteration Number	Constraints		Optimal Cost		
	Spec. ($\overline{Y}_i$)	Actual (Y_i)	J_{OCC}	Q_i	G^T
22	0.0350 0.0500 0.0500	0.0314 0.0126 0.0500	0.0340	0.0000 0.0000 2.3765	0.0193 -1.3839 -0.1374

Table 1: Solution to Problem 1 with $\alpha = 4.5$.

State Feedback Design					
Iteration Number	Constraints		Optimal Cost		
	Spec. ($\overline{Y}_i$)	Actual (Y_i)	J_{OCC}	Q_i	G^T
31	0.0300 $0.050 \times I_2$	0.0313 $\begin{bmatrix} 0.0123 & 0.0014 \\ 0.0014 & 0.0499 \end{bmatrix}$	0.0235	0.0000 $\begin{bmatrix} 0.0019 & 0.0527 \\ 0.0527 & 1.4277 \end{bmatrix}$	0.0212 -0.9542 -0.0950
Output Feedback Design					
Iteration Number	Constraints		Optimal Cost		
	Spec. ($\overline{Y}_i$)	Actual (Y_i)	J_{OCC}	Q_i	G^T
65	0.0350 $0.050 \times I_2$	0.0314 $\begin{bmatrix} 0.0126 & 0.0014 \\ 0.0014 & 0.0499 \end{bmatrix}$	0.0341	0.0000 $\begin{bmatrix} 0.0035 & 0.0919 \\ 0.0919 & 2.3809 \end{bmatrix}$	0.0149 -1.3878 -0.1379

Table 2: Solution to Problem 2 with $\alpha = 30$.

5 Conclusion

In this paper we have considered the so-called Output Covariance Constraint (OCC) control problem. This is the problem of minimizing control effort subject to matrix inequality constraints on several closed loop covariance matrices. Optimality conditions for characterizing a global solution are provided. In the state-feedback case, these conditions comprise one algebraic Riccati equation, one Lyapunov equation, and a coupling condition. The unknown in this system of equations is a matrix Q which may be interpreted as a matrix of Kuhn-Tucker multipliers. We have given an iterative algorithm to find such a matrix Q. Under the assumption that the optimality conditions admit a solution Q, we have shown that the iterative algorithm converges to one such solution, provided the step size parameter α is properly chosen. Using the separation property of a closed loop covariance matrix given in [12], we have shown how to modify the state-feedback algorithm to solve the OCC problem with output feedback. Both discrete and continuous time problems have been solved. Finally, an example is presented to demonstrate the applicability of our results.

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D. Covariance Active/Passive Control

COVARIANCE ACTIVE/PASSIVE CONTROL

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ABSTRACT

An integrated means for active controller design and structure redesign is presented. The techniques of covariance control are used to parametrize all possible combinations of active controllers/structure redesign parameters which can stabilize the plant, and achieve certain closed-loop performance.

1. INTRODUCTION

To cope with the demanding task of structural control problems, one can improve performance by combining both the design of active controllers and the design of the structure. The term "Smart Structures" implies the use of feedback control sensors and actuators imbedded in a material structure to improve the dynamic response. Until now, the design of the controller parameters and the structural material parameters have not been integrated and combined to guarantee any specific performance. Existing methods can choose the controller and structure parameters by trial and error or by gradient approaches to nonlinear programming problems.

These computationally intensive approaches are devoid of physical insight, do not guarantee stability, and make no attempt to produce all stabilizing solutions. Such is the goal of this paper, to show the explicit relationship between all stabilizing state feedback control gains and the structural parameters. If an initial structure is given, and if a controller which satisfies performance requirements (closed-loop stability, tracking accuracy... etc.) is given, the necessary and sufficient condition is known[6,7] for the existence of structure redesign parameters to duplicate the closed-loop system performance while minimizing the active control effort. This condition derived from the above setting is convex in the structural redesign parameters, and hence a global minimum is guaranteed, as well as stability.

The drawback in this past approach[6,7] is that we are given a controller before hand, hence the space in which we search for the optimum is necessarily restricted by this fact. In other words, it is possible that beginning with another controller, we can reduce the control effort even more. We seek to simultaneously redesign the structure and the controller. The above algorithm[6,7] does not necessarily solve this problem, even if applied iteratively.

To obviate this difficulty we will use the covariance control technique[8,2,5], which provides a way to parametrize all stabilizing controllers in terms of a physically meaningful state covariance X, and the stabilizability conditions derived in the theory for active control forms a parametrization of the set of all assignable covariances as function of structure parameters only (without the control

gains).

In this paper, we first present the problem formulation in section 2, and the main results are introduced in section 3. We give 2 examples in section 4, and in section 5, a short discussion of future directions is given. All proofs are given in appendices.

2. PROBLEM FORMULATION

We shall limit our attention to linear systems, although some advantages of our approach extend also to nonlinear systems. Assume that the equations of motion for the linear elastic structure have been put into the finite dimensional state form.

$$\dot{x} = Ax + Bu + Dw \tag{1}$$

$$u = G_a x \tag{2}$$

Where w(t) is a zero mean white noise (including actuator noise) with intensity W, and G_a is the state feedback control gain to be designed.

For a given state covariance $X > 0$, the necessary and sufficient condition for the existence of a G_a which assigns this X was derived by Yasuda and Skelton[8] as

$$(I - BB^+)Q(I - BB^+) = 0 \tag{3}$$
$$Q \triangleq AX + XA^* + DWD^*$$

and the set of G_a that satisfies the requirement is parametrized as:

$$G_a = -\frac{1}{2}B^+Q(2I - BB^+)X^{-1} + B^+SBB^+X^{-1} + (I - BB^+)Z \tag{4}$$

Since w includes actuator noise, then D has the structure $D = [B, D_2]$, and stabilizability (controllability) of (A,B) implies stabilizability (controllability) of (A,D). We assume controllability of (A,B) to simplify the presentation. If (A,B) is controllable, then $X > 0$ is equivalent to $(A + BG_a)$ stable. Hence, since (3) parametrizes all $X > 0$ that can be assigned to the system. Condition (3), with $X > 0$, is also a necessary and sufficient condition for stability of the closed loop system.

The significance of state covariance is well known. A system must be stable to have a bounded covariance, and almost all robustness properties of linear systems (disturbance rejection, structured and unstructured parameter robustness) can be related directly to properties of the state covariance[4,1].

Note that the control gain G in (4) is an explicit function of the covariance X and the plant (structure) data (A,B). If we can relate a set of closed-loop performance requirements to an $X > 0$, then all control gains that assign this X to the system are given by (4). Let A_0 denote the original structure, and G_p denote the changes in the structural parameters that are allowed. Now if we write A(system matrix) as

$$A = A_0 + B_p G_p M_p \tag{5}$$

The structure of the connectivity matrices B_p, M_p allow changes in the system matrix to be accomplished in a physically achievable way.

We call G_p the "passive controller." The closed-loop system matrix looks like so,

$$A_{cl} = A_0 + B_p G_p M_p + B G_a \ . \tag{6}$$

The question is, given a desired state covariance $X > 0$, is there a (G_p, G_a) pair to assign this X? This is equivalent to asking the stabilizability question, since every stable system has a finite positive $X > 0$. If it is stabilizable, we desire the set of all (G_p, G_a) which stabilize the system.

3. MAIN RESULT

Theorem 1 *A state feedback system*

$$\dot{x} = (A + B_p G_p M_p)x + Bu$$

$$u = G_a x$$

is stabilizable by some G_p, G_a iff there exists $X > 0$ satisfying

$$\left\{ \begin{array}{l} P_\beta PQPP_\beta = 0 \\ P_{\mathcal{M}} PQPP_{\mathcal{M}} = 0 \\ [(I - P_{\mathcal{M}}\beta\beta^+)(P_{\mathcal{M}}\beta\beta^+)^+]\, P_{\mathcal{M}} PQP = 0 \end{array} \right. \tag{7}$$

where

$$\begin{aligned} P &\triangleq (I - BB^+) \quad Q \triangleq (XA_0^* + A_0 X + DWD^*) \\ \beta &\triangleq PB_p \quad \mathcal{M} \triangleq M_p XP \\ P_\beta &\triangleq (I - \beta\beta^+), \quad P_{\mathcal{M}} \triangleq (I - \mathcal{M}^+\mathcal{M}) \end{aligned}$$

Proof. See Appendix A.

The conditions shown in (7) are similar to the covariance assignability conditions derived in[8] for the measurement feedback system. This similarily is due to the fact that the plant redesign part $B_p G_p M_p$ is mathematically equivalent to measurement feedback, and we take advantage of this fact in our proof for Theorem 1. Next, compare (3) (interpret with A_0 in lieu of A) with (5)-(7), we can see that the set of assignable X is enlarged because of the added flexibility of plant redesign. (Conditions (5)-(7) are less restrictive than (3). Any X which satisfies (3) will automatically satisfy (5)-(7), but the reverse is not true.)

The next theorem gives the parametrization of active and passive covariance controllers.

Theorem 2 *Suppose X is assignable. Then all (G_p, G_a) that assign X to the system are given by:*

$$\begin{aligned} G_a &= -\frac{1}{2}B^+[Q + B_pG_pM_pX + XM_p^*G_p^*B_p^* + P_a^+\Phi - (P_a^+\Phi)^* \\ &\quad -P_a^+\Phi P_a^+P_a]X^{-1} + B^+(I - P_a^+P_a)S_a(I - P_a^+P_a)^+ + P_aZ_a \qquad (8) \\ G_p &= -\frac{1}{2}\beta^+[PQP + L_p^+\Phi_p - (L_p^+\Phi_p)^* - L_p^+\Phi_pL_p^+L_p]\mathcal{M}^+ \\ &\quad +\beta^+(I - L_p^+L_p)S_p(I - L_p^+L_p)\mathcal{M}^+ + Z_p - \beta^+\beta Z_p\mathcal{M}\mathcal{M}^+ \qquad (9) \end{aligned}$$

Where

$$\Phi \triangleq -P[Q + B_pG_pM_pX + XM_p^*G_p^*B_p^*]$$

$$L_p \triangleq \begin{bmatrix} I - \beta\beta^+ \\ I - \mathcal{M}^+\mathcal{M} \end{bmatrix} \quad \Phi_p \triangleq \begin{bmatrix} -I + \beta\beta^+ \\ -I + \mathcal{M}^+\mathcal{M} \end{bmatrix} PQP$$

and β, P, Q, , $\mathcal{M}$, are defined as in Theorem 1. S_a, S_p are arbitrary skew-symmetric matrices, Z_a, Z_p are arbitrary matrices of proper dimension.

Proof. See Appendix B.

Note that we have 4 free parameters, i.e., (S_a, S_p, Z_a, Z_p) in the characterization of (G_p, G_a). These free parameters provide us with additional freedom for further optimization of a secondary objective (for example, searching for lowest fuel consumption or highest precision etc.), without changing the closed-loop state covariance.

In the following examples, we will show that with the additional freedom of plant redesign, we can achieve closed-loop performance which is not feasible with active control alone.

4. EXAMPLE

Example 1. Single Mass, Spring and Damper

The original system with active control only is

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw \\ u &= G_aM_ax \\ A &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \; M_a^T = B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad M_p^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

From (3), we can derive the assignable set of covariances as

$$X_a = \begin{bmatrix} a & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \; ; \; a > 0 \; .$$

where X_a satisfies

$$0 = X_a(A + BG_aM_a)^T + (A + BG_aM_a)X_a + DD^T \ .$$

With simultaneous plant/control design

$$\dot{x} = (A + B_pG_pM_p)x + B_au + Dw \ ,$$

the set of assignable closed loop covariances is

$$X = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \quad a_1 > 0, \ \ a_2 > 0.$$

where X satisfies

$$0 = X(A + B_pG_pM_p + B_aG_aM_a)^T + (A + B_pG_pM_p + B_aG_aM_a)X + DD^T \ .$$

It is obvious that the set of all assignable covariances by active control gain G_a is included in the set by the simultaneous plant/control design, i.e., the addition of plant redesign has enlarged the set of assignable covariances, and consequently, the closed-loop performance capability.

Example 2. Euler Bernoulli Beam, 2 modes.

Consider

$$\dot{x} = (A + B_pG_pM_p)x + B_au + Dw$$

where

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -0.01 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -16 & -0.04 \end{bmatrix} \quad B_a = D = \begin{bmatrix} 0 \\ 0.5878 \\ 0 \\ 0.955 \end{bmatrix} \\ M_p &= B_p^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

The choice of M_p, B_p corresponds to a damping mechanism to change the damping of each mode.

Given an assignable closed-loop covariance matrix:

$$X = \begin{bmatrix} 0.1183 & 0 & 0.0013 & -0.0296 \\ 0 & 0.1538 & 0.0296 & 0.047 \\ 0.0013 & 0.0296 & 0.0264 & 0 \\ -0.0296 & 0.047 & 0 & 0.4484 \end{bmatrix}$$

From Theorem 2, eq. (9), we compute

$$G_p = \begin{bmatrix} -0.7236 & 0.4472 \\ 0.4472 & -0.2764 \end{bmatrix} + \begin{bmatrix} 0.2764 & 0.4472 \\ 0.4472 & 0.7236 \end{bmatrix} Z_p \ .$$

Substitute G_p into G_a, then, from (8),

$$\begin{aligned} G_a &= [-0.5 \quad -0.4702 \quad -1 \quad -0.7608] + G_{a2}S_aG_{a3} \\ &\quad +G_{a4}Z_pG_{a5} + G_{a6}Z_p^TG_{a7} \end{aligned}$$

Where S_a is arbitrary skew symmetric and Z_p arbitrary.

$$\begin{aligned} G_{a2} &= [0 \quad 0.4702 \quad 0 \quad 0.7608] \\ G_{a3} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2248 & 1.9775 & -2.2228 & 0.8048 \\ 0 & 0 & 0 & 0 \\ 0.3637 & 3.1998 & -0.5967 & 1.3022 \end{bmatrix} \\ G_{a4} &= [-0.2351 \quad -0.3804] \\ G_{a5} &= \begin{bmatrix} -0.0517 & 1.5453 & 0.5111 & -0.1850 \\ -0.1737 & -1.5278 & 1.7173 & 1.3782 \end{bmatrix} \\ G_{a6} &= [-0.0547 \quad -0.1816] \\ G_{a7} &= \begin{bmatrix} 0.2248 & 1.9775 & -2.2228 & 0.8048 \\ 0.3637 & 3.1998 & -3.5967 & 1.3022 \end{bmatrix} \end{aligned}$$

Hence $G_p(Z_p)$ and $G_a(Z_p, S_a)$ form all possible combinations of active and passive controllers that are stabilizing.

5. CONCLUSION

All controllers (passive and state feedback active) that assign a specified covariance are parametrized. These parametrizations are expressed in terms of system matrices only. The next step will be to expand the theory to measurement feedback system and systems with dynamic controllers, and to apply the theory to specific smart structure problems.

6. APPENDIXES

Appendix A. Proof of Theorem 1.

For state feedback system (1)-(2) if $X > 0$ is assignable, we have

$$(I - BB^+)(XA^* + AX + DWD^*)(I - BB^+) = 0 \tag{10}$$

Now let $A = A_o + B_pG_pM_p$.

From (10). $\Rightarrow$

$$(I - BB^+)(X(A_o + B_pG_pM_p)^* + (A_o + B_pG_pM_p)X + DWD^*)(I - BB^+) = 0 \tag{11}$$

Let $P \triangleq (I - BB^+) \qquad Q \triangleq (XA_o^* + A_oX + DWD^*)$.

From (11). $\Rightarrow$

$$P(XM_p^*G_p^*B_p^* + B_pG_pM_pX)P = -PQP$$

$$\Rightarrow \quad PB_pG_pM_pXP = -\frac{1}{2}(PQP + S_k) \quad S_k^* = -S_k \tag{12}$$

Let $\beta \triangleq PB_p$, $\mathcal{M} \triangleq M_pXP$.

$$\Rightarrow \quad \beta G_p\mathcal{M} = -\frac{1}{2}(PQP + S_k) \tag{13}$$

For $G_p\mathcal{M}$ to have solution:

$$(I - \beta\beta^+)(PQP + S_k) = 0 \tag{14}$$

For βG_p to have solution

$$(PQP + S_k)(I - \mathcal{M}^+\mathcal{M}) = 0 \tag{15}$$

It is easy to prove that (14), (15) are necessary and sufficient conditions for the existence of G_p. Next, solve S_k, because S_k is skew-symmetric.

$\Rightarrow$ jS_k is Hermitian.

$$From\ (14) \ \Rightarrow \ (I - \beta\beta^+)(jS_k) = -j(I - \beta\beta^+)PQP \tag{16}$$

$$From\ (15) \ \Rightarrow \ (jS_k)(I - \mathcal{M}^+\mathcal{M}) = -jPQP[I - \mathcal{M}^+\mathcal{M}] \tag{17}$$

We need the following lemma:

Lemma 1 $AX = C$, $XB = D$ *has common Hermitian Solution* X *is and only if these two equations have common solution and*

$$H^* = H \ ; \ H = \begin{bmatrix} CA^* & CB \\ D^*A^* & D^*B \end{bmatrix} \ .$$

Now, if we set $A = (I - \beta\beta^+)$, $B = (I - \mathcal{M}^*\mathcal{M})$

$$C = -j(I - \beta\beta^+)PQP, \quad D = -JPQP[I - \mathcal{M}^*\mathcal{M}]$$

then, from lemma 1, we can see that if jS_k *has Hermitian solution*

$$CA^* = AC^* \ \Rightarrow \ [I - \beta\beta^+]PQP[I - \beta\beta^+] = 0 \tag{18}$$

$$D^*B = B^*D \ \Rightarrow \ [I - \mathcal{M}^*\mathcal{M}]PQP(I - \mathcal{M}^*\mathcal{M}) = 0 \ . \tag{19}$$

Notice that

$$(17) \Rightarrow \ (I - \mathcal{M}^+\mathcal{M})^*(jS_k)^* = j(I - \mathcal{M}^+\mathcal{M})PQP \tag{20}$$

$$(16),(20) \Rightarrow \begin{cases} (I - \beta\beta^+)(jS_k) = -j(I - \beta\beta^+)PQP \\ (I - \mathcal{M}^+\mathcal{M})(jS_k) = j(I - \mathcal{M}^+\mathcal{M})PQP \end{cases}$$

We require (16), (20) to possess common solutions.

It is obvious that (16) always has solution:

$$(jS_k) = -jP_\beta PQP - (I - P_\beta)Z \tag{21}$$

Substitute (21) into (20)

$$\Rightarrow \quad -P_m(I - P_\beta)Z = jP_m(I - P_\beta)PQP \tag{22}$$

(22) is solvable iff

$$[I - (P_m\beta\beta^+)(P_m\beta\beta^+)^+]P_m(I + P_\beta)PQP = 0$$

$$\Rightarrow [I - (P_m\beta\beta^+)(P_m\beta\beta^+)^+]P_mPQP = 0 \tag{23}$$

Hence, we have proved that (14), (15) are equivalent to (18), (19), (23), i.e., for G_p to be solvable iff (18), (19), (23) hold.

Q.E.D.

Appendix B. Proof of Theorem 2

We need an extension of lemma 1.

Lemma 2 [3]

Suppose the equation $AX = C$, $XB = D$ have a common Hermitian solution, the form of the solution is

$$\begin{aligned} X \;=\; & \begin{bmatrix} A \\ B^* \end{bmatrix}^- \begin{bmatrix} C \\ D^* \end{bmatrix} + \begin{bmatrix} C \\ D^* \end{bmatrix}^* \left[\begin{bmatrix} A \\ B^* \end{bmatrix}^-\right]^* - \begin{bmatrix} A \\ B^* \end{bmatrix}^- \begin{bmatrix} C \\ D^* \end{bmatrix} \begin{bmatrix} A \\ B^* \end{bmatrix}^* \\ & \left[\begin{bmatrix} A \\ B^* \end{bmatrix}^*\right]^* + \left[I - \begin{bmatrix} A \\ B^* \end{bmatrix}^- \begin{bmatrix} A \\ B^* \end{bmatrix}\right] U \left[I \begin{bmatrix} A \\ B^* \end{bmatrix}^- \begin{bmatrix} A \\ B^* \end{bmatrix}\right] \end{aligned} \tag{24}$$

Where U is an arbitrary Hermitian matrix and $[\cdot]^-$ is the generalized inverse.

To generate all solutions, we must parameterize all pseudo inverse $[\cdot]^-$ as follows

$$\begin{bmatrix} A \\ B^* \end{bmatrix}^- = \begin{bmatrix} A \\ B^* \end{bmatrix}^- + Z - \begin{bmatrix} A \\ B^* \end{bmatrix}^+ \begin{bmatrix} A \\ B^* \end{bmatrix} Z \begin{bmatrix} A \\ B^* \end{bmatrix} \begin{bmatrix} A \\ B^* \end{bmatrix}^+ \tag{25}$$

where Z is arbitrary, $[\cdot]^+$ denotes the Moore Penrose inverse.

Now, if we define

$$L_p \triangleq \begin{bmatrix} I - \beta\beta^+ \\ I - M^+M \end{bmatrix} \quad \Phi_p \triangleq \begin{bmatrix} -(I - \beta\beta^+) \\ -(I - M^+M) \end{bmatrix} PQP$$

then according to (24), (25),

$$S_k = L_p^+\Phi_p + (L_p^+\Phi_p)^* - L_p^+\Phi_pL_p^+L_p + (I - L_p^+L_p)\hat{S}(Z - L_p^+L_p) \tag{26}$$

$\hat{S}$ is an arbitrary skew-symmetric matrix. Next, plug (26) into standard form of solution for (13), we get:

$$
\begin{aligned}
G_p &= -\frac{1}{2}\beta^+[PQP + L_p^+\Phi_p - (L_p^+\Phi_p)^* - L_p^+\Phi_p L_p^* L_p]M^+ \\
&\quad +\beta^+(I - L_p^+L_p)S_p(I - L_p^+L_p)M^+ + Z_p - \beta_+\beta Z_p MM^+
\end{aligned}
$$

where Z_p is arbitrary.

The parametrization of G_a was derived in[7].

Q.E.D.

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E. On the Observer-based Structure of Covariance Controllers

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On the observer-based structure of covariance controllers

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Abstract: The problem of finding the set of all stabilizing full-order controllers is equivalent to a problem of finding all matrices which can be assigned to the closed-loop system as a state covariance. The necessary and sufficient conditions for a given matrix to be assignable as a covariance are given, and all controllers (of plant order) which assign a specified covariance are parametrized explicitly. The structure of covariance controllers is shown for the first time to be observer-based (state estimator plus estimated-state feedback). The 'central' state estimator of the covariance controller is shown to be the Kalman filter. Unlike the traditional estimator-based controller, the separation principle does not hold, but one can design a controller by assigning the estimation error covariance and the plant state covariance simultaneously.

Keywords: Covariance control; all stabilizing controllers; observer; Lyapunov method; separation principle.

1. Introduction

In practical controller design problems, it is frequently required to impose performance specifications on the variances of *each* output. Covariance control theory was originally motivated by such multiple design objectives: the multiple inequality constraints can be satisfied by assigning a matrix (which carries the specified variance value of each state on the diagonal) to the closed-loop system as the state covariance. Along with the development of covariance control theory [4–9, 11–18] many additional advantages have been investigated. Corless et al. [1] showed that the state covariance is explicitly related to such disturbance robustness properties as the $\mathscr{L}_2$ to $\mathscr{L}_\infty$ gain of the system, and such stability robustness properties as an upper bound on the stabilizing perturbations in the system matrix, etc. Yasuda et al. [17] showed that the covariance controller is a parametrization of the set of all stabilizing fixed-order controllers. It has also been shown [15] that covariance controllers are useful for designing reduced-order controllers.

In this paper, we consider the covariance assignment problem with full-order controllers (of order equal to the plant) for linear time-invariant continuous-time plants. The necessary and sufficient conditions for a matrix to be assignable as a state covariance are obtained and explicit controller formulas for all full-order controllers which assign a specified covariance are given. These new results are obtained for the plant with noisy measurements (most of the covariance control literature assumes noise-free measurements, see [9, 12] for a few exceptions). Based on these results, the structure of covariance controllers is studied from the point of view of estimation error covariance and plant state covariance assignments. Design of observers with estimation error covariance assignment was discussed in [18] and connected in an ad hoc way to an estimated-state feedback for controller design. Unlike [18], the present paper reveals an observer-based structure in the covariance controller without any assumptions. We shall show that the separation principle does not hold for covariance controllers, and that the state estimator and the estimated-state feedback gain

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cannot be designed independently. Nevertheless, the estimation error covariance and the plant state covariance to be assigned can be specified subject to an inequality constraint on these two covariances. Such controller design methods based on the covariance assignment are computationally tractable since an assignable estimation error covariance can be obtained by solving the LQG-type Riccati equation, and an assignable plant state covariance can be found as a solution to a finite-dimensional convex feasibility problem [4].

The paper is organized as follows. Section 2 describes the system models which we will consider and poses the covariance control problem. Section 3 presents new assignability conditions for closed-loop state covariance and explicit controller formulas. Section 4 introduces the concept of plant state covariance assignment and studies the structure of covariance controllers. Conclusions are given in Section 5.

We will use the following notation. $\mathscr{E}[\cdot]$ denotes the expectation operator. For a matrix A, A^{T} denotes the transpose of A. A^{+} is the Moore–Penrose inverse of A. For a nonnegative definite matrix A, $A^{1/2}$ denotes the unique nonnegative definite square root of A. $\mathrm{tr}[A]$ is the trace of A. $\mathrm{rank}[A]$ is the rank of A. SVD stands for the singular value decomposition.

2. The covariance control problem

We consider the following linear time-invariant continuous-time system (Σ_{p}):

$$\dot{x}_{\mathrm{p}} = A_{\mathrm{p}}x_{\mathrm{p}} + B_{\mathrm{p}}u + D_{\mathrm{p}}w_{\mathrm{p}}, \tag{2.1}$$

$$z = M_{\mathrm{p}}x_{\mathrm{p}} + v, \tag{2.2}$$

where $w_{\mathrm{p}} \in \Re^{n_w}$ and $v \in \Re^{n_z}$ are zero mean white noise stochastic processes with intensities $W_{\mathrm{p}} > 0$ and $V > 0$, respectively, and correlation $\mathscr{E}[w_{\mathrm{p}}(t)v^{\mathrm{T}}(\tau)] = W_{12}\delta(t-\tau)$. $x_{\mathrm{p}} \in \Re^{n_{\mathrm{p}}}$ is the plant state, $u \in \Re^{n_u}$ is the control input, and $z \in \Re^{n_z}$ is the available measurement. Throughout the paper, we will assume that $(A_{\mathrm{p}}, D_{\mathrm{p}})$ is controllable, $(A_{\mathrm{p}}, B_{\mathrm{p}})$ is stabilizable, and $(A_{\mathrm{p}}, M_{\mathrm{p}})$ is detectable. Consider a full-order dynamic controller (Σ_{c}):

$$\dot{x}_{\mathrm{c}} = A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}z, \tag{2.3}$$

$$u = C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c}}z, \tag{2.4}$$

where $x_{\mathrm{c}} \in \Re^{n_{\mathrm{p}}}$ is the controller state. Combining (Σ_{c}) and (Σ_{p}), the closed-loop system (Σ_{cl}) can be described by

$$\dot{x} = A_{\mathrm{cl}}x + B_{\mathrm{cl}}w, \tag{2.5}$$

where

$$x := \begin{bmatrix} x_{\mathrm{p}} \\ x_{\mathrm{c}} \end{bmatrix}, \quad w := \begin{bmatrix} w_{\mathrm{p}} \\ v \end{bmatrix}, \quad A_{\mathrm{cl}} := \begin{bmatrix} A_{\mathrm{p}} + B_{\mathrm{p}}D_{\mathrm{c}}M_{\mathrm{p}} & B_{\mathrm{p}}C_{\mathrm{c}} \\ B_{\mathrm{c}}M_{\mathrm{p}} & A_{\mathrm{c}} \end{bmatrix}, \quad B_{\mathrm{cl}} := \begin{bmatrix} D_{\mathrm{p}} & B_{\mathrm{p}}D_{\mathrm{c}} \\ 0 & B_{\mathrm{c}} \end{bmatrix}. \tag{2.6}$$

It can be shown that the closed-loop state covariance

$$X := \mathscr{E}[x(t)x(t)^{\mathrm{T}}] = \begin{bmatrix} X_{\mathrm{p}} & X_{\mathrm{pc}} \\ X_{\mathrm{pc}}^{\mathrm{T}} & X_{\mathrm{c}} \end{bmatrix} \tag{2.7}$$

satisfies the following Lyapunov equation

$$Q := A_{\mathrm{cl}}X + XA_{\mathrm{cl}}^{\mathrm{T}} + B_{\mathrm{cl}}WB_{\mathrm{cl}}^{\mathrm{T}} = 0, \tag{2.8}$$

where W is the noise intensity defined by

$$\mathscr{E}[w(t)w(\tau)^{\mathrm{T}}] =: W\delta(t-\tau) = \begin{bmatrix} W_{\mathrm{p}} & W_{12} \\ W_{12}^{\mathrm{T}} & V \end{bmatrix}\delta(t-\tau). \tag{2.9}$$

Definition 1. A positive-definite matrix X is said to be *assignable* as a closed-loop state covariance if it satisfies (2.8) for some controller (Σ_{c}).

Since (A_p, D_p) is controllable, it can easily be shown (see e.g. [7]) that (A_{cl}, B_{cl}) is also controllable for any choice of (Σ_c) provided the controller is minimal, i.e., (A_c, B_c, C_c) is controllable and observable. Therefore, by Lyapunov theory, A_{cl} is asymptotically stable (all its eigenvalues lie in the open left half plane of the complex plane) iff $X > 0$. Thus, by assigning only positive-definite state covariances, closed-loop stability is guaranteed, and conversely, the set of all stabilizing controllers is captured, since every stable system has a finite covariance.

Throughout the paper, we assume that the plant state and the controller state are fully correlated, i.e., X_{pc} has full rank. The assumption is not restrictive, since it is necessary to guarantee the 'minimality' of the controller in the sense that there is no controller of smaller order which yields the same plant state covariance, see [8] for more general and detailed treatment of this minimality concept. A technical implication of the assumption here is that X_{pc} is invertible since the controller order is equal to that of the plant and X_{pc} is a square invertible matrix. Indeed, it has been shown in [3] that if X_{pc} is not invertible, then a controller state variable can be deleted without changing the closed-loop plant covariance.

Our objective is to design a controller which yields a specified matrix $X > 0$ as the closed-loop state covariance. In order to accomplish this objective, we pose the following questions:

(i) What are the necessary and sufficient conditions for assignability?

(ii) What are the controllers which assign a given assignable covariance?

Answers to questions (i) and (ii) are given in the next section.

3. Assignability conditions and controller formulas

The following theorem yields necessary and sufficient conditions for a positive-definite matrix to be assignable as the closed-loop state covariance and gives explicit formulas for all (full-order) controllers (Σ_c) which assign a given state covariance $X > 0$.

Theorem 3.1. *Let a positive-definite matrix $X \in \Re^{2n_p \times 2n_p}$ be given. Then X is assignable as a closed-loop state covariance iff it satisfies*

$$(I - B_pB_p^+)(A_pX_p + X_pA_p^T + D_pW_pD_p^T)(I - B_pB_p^+) = 0, \tag{3.1}$$

$$R := -A_p\bar{X}_p - \bar{X}_pA_p^T - D_pW_pD_p^T + (\bar{X}_pM_p^T + D_pW_{12})V^{-1}(\bar{X}_pM_p^T + D_pW_{12})^T \geq 0, \tag{3.2}$$

$$\operatorname{rank}(R) \leq n_z, \tag{3.3}$$

where

$$\bar{X}_p := X_p - X_{pc}X_c^{-1}X_{pc}^T. \tag{3.4}$$

If these conditions hold, all controllers which assign X are given by

$$A_c = X_cX_{pc}^{-1}(A_p + B_pD_cM_p + L_pV^{1/2}B_c^TX_{pc}^{-1})X_{pc}X_c^{-1} - B_cM_pX_{pc}X_c^{-1} + X_cX_{pc}^{-1}B_pC_c, \tag{3.5}$$

$$B_c = X_cX_{pc}^{-1}[(\bar{X}_pM_p^T + D_pW_{12})V^{-1} + B_pD_c - L_pV^{-1/2}], \tag{3.6}$$

$$\begin{aligned} C_c = {}& \tfrac{1}{2}B_p^+(Q_p + B_pD_cVD_c^TB_p^T + B_pD_c\Gamma_p^T + \Gamma_pD_c^TB_p^T)(2I - B_pB_p^+)X_{pc}^{-T} \\ & + B_p^+SB_pB_p^+X_{pc}^{-T} + (I - B_p^+B_p)Z, \end{aligned} \tag{3.7}$$

$$D_c = \textit{arbitrary}, \tag{3.8}$$

where Z is arbitrary and S is an arbitrary skew-symmetric matrix and

$$R := L_pL_p^T, \quad L_p \in \Re^{n_p \times n_z}, \tag{3.9}$$

$$Q_p := A_pX_p + X_pA_p^T + D_pW_pD_p^T, \qquad \Gamma_p := X_pM_p^T + D_pW_{12}. \tag{3.10}$$

Proof. Computing each partitioned block of Q defined by (2.8), we have

$$Q_{11} := Q_p + B_p D_c V D_c^T B_p^T + B_p D_c \Gamma_p^T + \Gamma_p D_c^T B_p^T + B_p C_c X_{pc}^T + X_{pc} C_c^T B_p^T = 0, \tag{3.11}$$

$$Q_{12} := (A_p + B_p D_c M_p) X_{pc} + X_{pc} A_c^T + (\Gamma_p + B_p D_c V) B_c^T + B_p C_c X_c = 0, \tag{3.12}$$

$$Q_{22} := A_c X_c + X_c A_c^T + B_c M_p X_{pc} + X_{pc}^T M_p^T B_c^T + B_c V B_c = 0, \tag{3.13}$$

where Q_p and Γ_p are defined by (3.10).

Necessity: Suppose a given matrix $X > 0$ is assignable as a state covariance. Then there exists a controller (A_c, B_c, C_c, D_c) satisfying (3.11)–(3.13). Pre- and post-multiplying (3.11) by $(I - B_p B_p^+)$ immediately yields (3.1). Let

$$P_x := [I \quad -X_{pc} X_c^{-1}], \qquad \bar{Q} := P_x Q P_x^T. \tag{3.14}$$

Using (3.11)–(3.13), (3.14) yields

$$\bar{Q} = \bar{Q}_p + (B_p D_c - X_{pc} X_c^{-1}) \bar{\Gamma}_p^T + \bar{\Gamma}_p (B_p D_c - X_{pc} X_c^{-1} B_c)^T$$
$$+ (B_p D_c - X_{pc} X_c^{-1} B_c) V (B_p D_c - X_{pc} X_c^{-1} B_c)^T = 0 \tag{3.15}$$

where

$$\bar{Q}_p := A_p \bar{X}_p + \bar{X}_p A_p^T + D_p W_p D_p^T, \qquad \bar{\Gamma}_p := \bar{X}_p M_p^T + D_p W_{12}. \tag{3.16}$$

Completing the square, (3.15) is equivalent to

$$\bar{Q}_p - \bar{\Gamma}_p V^{-1} \bar{\Gamma}_p^{-T} + L_p L_p^T = 0, \tag{3.17}$$

where

$$L_p := (B_p D_c - X_{pc} X_c^{-1} B_c + \bar{\Gamma}_p V^{-1}) V^{1/2}. \tag{3.18}$$

Since $L_p L_p^T \geq 0$ and rank $(L_p L_p^T) \leq n_z$, (3.17) shows that the conditions (3.2) and (3.3) are necessary.

Sufficiency: Suppose the assignability conditions (3.1)–(3.3) are satisfied for a matrix $X > 0$. Sufficiency will be shown by constructing all controller matrices satisfying (3.11)–(3.13).

Lemma 3.1 (Skelton and Ikeda [11]). *Let matrices A, B, X and Q be given where X is invertible and $Q = Q^T$. Then there exists a matrix G such that*

$$BGX + (BGX)^T + Q = 0, \tag{3.19}$$

iff

$$(I - BB^+) Q (I - BB^+) = 0 \tag{3.20}$$

holds, in which case, all such matrices G are given by

$$G = -\tfrac{1}{2} B^+ Q (2I - BB^+) X^{-1} + B^+ S B B^+ X^{-1} + (I - B^+ B) Z, \tag{3.21}$$

where Z is arbitrary and S is an arbitrary skew-symmetric matrix.

Applying Lemma 3.1 to equation (3.11), the existence of C_c satisfying $Q_{11} = 0$ is guaranteed (for any choice of D_c) by (3.1) and invertibility of X_{pc}, and all such matrices C_c are given by

$$C_c = \tfrac{1}{2} B_p^+ (Q_p + B_p D_c V D_c^T B_p^T + B_p D_c \Gamma_p^T + \Gamma_p D_c^T B_p^T)(2I - B_p B_p^+) X_{pc}^{-T}$$
$$+ B_p^+ S B_p B_p^+ X_{pc}^{-T} + (I - B_p^+ B_p) Z, \tag{3.22}$$

where Z is arbitrary and S is an arbitrary skew-symmetric matrix. Now, instead of solving (3.12) and (3.13), we consider the following equivalent equation:

$$\begin{bmatrix} I & -X_{pc}X_c^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix} = 0. \tag{3.23}$$

The first row block of (3.23) gives

$$(A_p + B_pD_cM_p - X_{pc}X_c^{-1}B_cM_p)X_{pc} + B_pC_cX_c - X_{pc}X_c^{-1}A_cX_c + L_pV^{1/2}B_c^T = 0. \tag{3.24}$$

Solving for A_c, we have

$$A_c = X_cX_{pc}^{-1}(A_p + B_pD_cM_p + L_pV^{1/2}B_c^TX_{pc}^{-1})X_{pc}X_c^{-1} - B_cM_pX_{pc}X_c^{-1} + X_cX_{pc}^{-1}B_pC_c. \tag{3.25}$$

Finally, we need to show the existence of B_c and D_c satisfying the second row block of (3.23), or $Q_{22} = 0$, with A_c and C_c given by (3.25) and (3.22), respectively. Recall that, if any solution B_c exists, then it must satisfy (3.18) for some L_p where L_p satisfies (3.17). Solving (3.18) for B_c,

$$B_c = X_cX_{pc}^{-1}(\bar{\Gamma}_pV^{-1} + B_pD_c - L_pV^{-1/2}). \tag{3.26}$$

Now we claim that, given any matrix D_c, the controller matrices A_c, B_c and C_c given by (3.25), (3.26) and (3.22) satisfy $Q_{22} = 0$. This can be verified as follows. Substituting A_c and B_c into (3.13), then using (3.11) to eliminate C_c, we have

$$Q_{22} = -X_cX_{pc}^{-1}(\bar{Q}_p - \bar{\Gamma}_pV^{-1}\bar{\Gamma}_p^{-T} + L_pL_p^T)X_{pc}^{-T}X_c = 0, \tag{3.27}$$

where the last equality holds due to (3.17). □

Theorem 3.1 provides an easy way to *check* if a specified matrix $X > 0$ is assignable as a state covariance. However, a desirable state covariance (determined by design specifications) is not assignable in most cases. Hence, we need a method to *construct* an assignable covariance. The following corollary gives an alternative characterization of the set of all assignable state covariances, which is useful for constructing an assignable covariance. Hereafter, we will assume that there is no correlation between the process noise w_p and the measurement noise v, i.e., $W_{12} = 0$ for simplicity.

Corollary 3.1. *Let X_p and $\bar{X}_p$ be any matrices satisfying*

$$(I - B_pB_p^+)(A_pX_p + X_pA_p^T + D_pW_pD_p^T)(I - B_pB_p^+) = 0, \tag{3.28}$$

$$A_p\bar{X}_p + \bar{X}_pA_p^T - \bar{X}_pM_p^TV^{-1}M_p\bar{X}_p + D_pW_pD_p^T + L_pL_p^T = 0, \tag{3.29}$$

$$0 < \bar{X}_p < X_p \tag{3.30}$$

for some $L_p \in \Re^{n_p \times n_z}$. Then the set of all assignable covariances can be generated by

$$X = \begin{bmatrix} X_p & NT^T \\ TN^T & TT^T \end{bmatrix} \tag{3.31}$$

where T is any nonsingular matrix and

$$N := (X_p - \bar{X}_p)^{1/2}. \tag{3.32}$$

Proof. Noting that (3.2) and (3.3) hold iff $R = L_pL_p^T$ for some $L_p \in \Re^{n_p \times n_z}$, we see that any assignable covariance X must satisfy (3.28) and (3.29) for some L_p. It is necessary that $\bar{X}_p > 0$ since $X > 0$, and $X_p - \bar{X}_p = X_{pc}X_c^{-1}X_{pc}^T > 0$ since $X_c > 0$ and X_{pc} is a square invertible matrix. This proves the necessity of (3.28)–(3.30). Sufficiency can easily be verified by noting that $NT^T = X_{pc}$ and $TT^T = X_c$. □

The above results are useful for *construction* of assignable state covariances since (3.28) and (3.29) are uncoupled in terms of X_p and $\bar{X}_p$, and hence X_p and $\bar{X}_p$ can be obtained by solving the linear equation (3.28) for $X_p > 0$ and the standard Riccati equation (3.29) for $\bar{X}_p > 0$. However, we must still check if $X_p > \bar{X}_p$ as in (3.30).

When constructing an assignable state covariance as in (3.31) for given X_p and $\bar{X}_p$ satisfying (3.28)–(3.30), we have a freedom in choosing a nonsingular matrix T. However, this freedom does not affect the closed-loop performance since the matrix T contributes only to the similarity transformation on the controller coordinate. Thus, the matrices X_p and $\bar{X}_p$ are the only factors that determine the closed-loop performance embedded in the state covariance X. In view of the controller formula (3.5)–(3.8), the other freedoms for the closed-loop performance are the direct measurement feed through gain D_c and the arbitrary matrix Z and the arbitrary skew-symmetric matrix S. The freedom Z disappears if there is no redundant actuators, i.e., $B_p^T B_p > 0$, since in this case $I - B_p^+ B_p = 0$. The skew-symmetric freedom S can be utilized for minimization of the input energy by the similar method to that of [13].

4. The structure of covariance controllers

In this section, the structure of covariance controllers is studied. It will be shown that the full-order strictly proper covariance controller can be interpreted as an observer-based controller, i.e., a state estimator plus an estimated-state feedback. To this end, we restrict our attention to strictly proper controllers and hence to a special choice of the arbitrary D_c in Theorem 3.1 ($D_c = 0$).

As has been shown in Corollary 3.1, the controller state covariance X_c and the correlation X_{pc} are controller-coordinate-dependent. Moreover, in general, the description of output performance specifications, such as the variances of each output, does not require the whole closed-loop state covariance X but only the *plant* state covariance X_p. Hence, assigning a matrix X_p as the plant state covariance makes more sense than the closed-loop state covariance. This motivates the plant state covariance assignment problem, defined as follows.

Definition 2. A positive-definite matrix $X_p \in \Re^{n_p \times n_p}$ is said to be *assignable* as a *plant* state covariance if there exists a controller which yields $\mathscr{E}[x_p(t)x_p(t)^T] = X_p$.

As in the case of the closed-loop state covariance assignment problem, the solution to the plant state covariance assignment problem consists of (1) necessary and sufficient conditions for a matrix X_p to be assignable as a plant state covariance, and (2) an explicit formula for all controllers which assign a specified plant covariance. The following theorem gives not only the solution but also a transparent description of the covariance controller structure.

Theorem 4.1. *Let a positive-definite matrix $X_p \in \Re^{n_p \times n_p}$ be given. Then X_p is assignable as a plant state covariance iff it satisfies*

$$(I - B_p B_p^+)(A_p X_p + X_p A_p^T + D_p W_p D_p^T)(I - B_p B_p^+) = 0, \tag{4.1}$$

$$X_p > P, \tag{4.2}$$

where P is the unique positive-definite solution to the following standard Riccati equation:

$$A_p P + P A_p^T - P M_p^T V^{-1} M_p P + D_p W_p D_p^T = 0, \tag{4.3}$$

in which case, all strictly proper full-order controllers which assign X_p are given by

$$A_c = A_p - B_c M_p + B_p C_c + L_p V^{1/2} B_c^T \bar{X}_c^{-1}, \tag{4.4}$$

$$B_c = \bar{X}_p M_p^T V^{-1} - L_p V^{-1/2}, \tag{4.5}$$

$$C_c = -\tfrac{1}{2} B_p^+ (A_p X_p + X_p A_p^T + D_p W_p D_p^T)(2I - B_p B_p^+)\bar{X}_c^{-1} + B_p^+ S B_p B_p^+ \bar{X}_c^{-1} + (I - B_p^+ B_p) Z, \tag{4.6}$$

where Z is arbitrary and S is an arbitrary skew-symmetric matrix and $L_p \in \Re^{n_p \times n_z}$ is any matrix such that

$$\bar{X}_c := X_p - \bar{X}_p > 0, \tag{4.7}$$

where $\bar{X}_p$ is the unique positive-definite solution $\bar{X}_p > 0$ of

$$A_p\bar{X}_p + \bar{X}_pA_p^T - \bar{X}_pM_p^TV^{-1}M_p\bar{X}_p + D_pW_pD_p^T + L_pL_p^T = 0, \tag{4.8}$$

where the controller coordinate has been chosen ($T = N$ in (3.31), (3.32)) so that the resulting closed-loop state covariance is given by

$$X := \begin{bmatrix} X_p & \bar{X}_c \\ \bar{X}_c & \bar{X}_c \end{bmatrix}. \tag{4.9}$$

Proof. Suppose $X_p > 0$ is assignable as a plant state covariance. Then from Corollary 3.1, X_p must satisfy (3.28) and (3.30) for some $\bar{X}_p > 0$ which solves the Riccati equation (3.29). From the standard monotonicity property of the solution to the Riccati equation (see e.g. [10]), the stabilizing solution $\bar{X}_p$ satisfies

$$0 < P \le \bar{X}_p \tag{4.10}$$

for any choice of L_p where P is the stabilizing solution for the case $L_p = 0$. This proves the necessity. To prove sufficiency, suppose $X_p > 0$ satisfies (4.1) and (4.2). Then there exists a matrix $\bar{X}_p > 0$ satisfying (3.29) and (3.30) for some (small enough) L_p, which can be verified by a choice $L_p = 0$. Hence, we can construct an assignable closed-loop state covariance X as in (3.31) and compute the controller matrices using (3.5)–(3.8), where we choose $D_c = 0$ since only the strictly proper controller is of interest here. This proves the sufficiency. Finally, the controller formula (4.4)–(4.6) can be obtained by choosing the controller-coordinate transformation matrix T in (3.31) to be N, defined by (3.32). □

Note that the set of all assignable plant state covariances is convex as the intersection of the linear affine set defined by (4.1) and the convex cone defined by (4.2). Thus, the controller design based on the plant state covariance assignment is computationally tractable (see [4]).

To interpret Theorem 4.1 in terms of the concept of observer-based controllers, we refer to the following results on the state estimation with error covariance assignment [18] and the covariance control with state feedback [5, 11].

Theorem 4.2 (Yaz and Skelton [18]). *Given a linear time-invariant continuous-time plant (Σ_p), consider the following state estimator (Σ_{se}):*

$$\dot{\hat{x}}_p = A_p\hat{x}_p + B_pu + F(z - M_p\hat{x}_p), \tag{4.11}$$

where the estimator gain F is the only design parameter. There exists a state estimator (Σ_{se}) which assigns a given matrix $E > 0$ as the estimation error covariance, i.e.,

$$E = \mathscr{E}[(x_p - \hat{x}_p)(x_p - \hat{x}_p)^T], \tag{4.12}$$

iff E satisfies the Riccati equation

$$A_pE + EA_p^T - EM_p^TV^{-1}M_pE + D_pW_pD_p^T + L_pL_p^T = 0 \tag{4.13}$$

for some $L_p \in \Re^{n_p \times n_z}$, in which case, all estimator gains which yield (4.12) *are given by*

$$F = EM_p^TV^{-1} - L_pV^{-1/2}. \tag{4.14}$$

Theorem 4.3 (Hotz and Skelton [5] and Skelton and Ikeda [11]). *Consider a linear time-invariant continuous-time plant (Σ_p). A positive-definite matrix $X_p > 0$ is assignable as a state covariance via state feedback control law $u = Gx_p$ iff X_p satisfies*

$$(I - B_pB_p^+)(A_pX_p + X_pA_p^T + D_pW_pD_p^T)(I - B_pB_p^+) = 0, \tag{4.15}$$

and the set of all state feedback gains G which assign X_p is given by

$$G = -\tfrac{1}{2} B_p^+(X_p A_p^T + A_p X_p + D_p W_p D_p^T)(2I - B_p B_p^+) X_p^{-1} + B_p^+ S B_p B_p^+ X_p^{-1} + (I - B_p^+ B_p) Z, \quad (4.16)$$

where Z is arbitrary and S is an arbitrary skew-symmetric matrix.

In view of Theorems 4.2 and 4.3, observer-based feedback structure of the full-order covariance controller described by Theorem 4.1 is evident by comparing (4.5) with (4.14) and (4.6) with (4.16). We will discuss the structure for the state estimator part and the estimated-state feedback part separately.

First consider the state estimator part. Note that the set of all B_c (with freedom L_p) given by (4.5) and (4.8) is exactly the same as the set of all admissible estimator gains F given by (4.14) and (4.13) where an estimator gain F is said to be 'admissible' if it yields a finite error covariance. Comparing (4.8) with (4.13), we can attach a physical significance to $\bar{X}_p$ (which has been defined as the Schur complement of X in (3.4)) as the estimation error covariance. In fact, using (4.7) and (4.9),

$$\mathscr{E}[(x_p - x_c)(x_p - x_c)^T] = \mathscr{E}[x_p x_p^T] - \mathscr{E}[x_p x_c^T] - \mathscr{E}[x_c x_p^T] + \mathscr{E}[x_c x_c^T]$$
$$= X_p - \bar{X}_c - \bar{X}_c + \bar{X}_c = \bar{X}_p.$$

Also note that, from (4.4), (2.3), and (2.4)

$$\dot{x}_c = A_p x_c + B_p u + B_c(z - M_p x_c) + L_p V^{1/2} B_c^T \bar{X}_c^{-1} x_c \quad (4.17)$$

which has an extra term due to L_p (the last term) when compared with (4.11). If we call the estimator part of the covariance controller obtained by choosing $L_p = 0$ the 'central estimator', it is apparent that the central estimator is the Kalman filter. Since $P > 0$ given by (4.3) satisfies $P \le \bar{X}_p$ for any choice of L_p, the Kalman filter optimizes not only the scalar objective $\mathrm{tr}(\bar{X}_p)$ as in the standard LQG theory but also the matrix-valued, or multiobjective function $\bar{X}_p$ (in the sense $P \le \bar{X}_p$ over all L_p). On the other hand, nonzero choices of the free matrix L_p can improve some other performances. For instance, it has been shown [2] that the central $\mathscr{H}_\infty$ estimator has the following structure:

$$\dot{x}_c = A_p x_c + B_p u + B_c(z - M_p x_c) + D_p \hat{w}_{\text{worst}}, \quad (4.18)$$

where $\hat{w}_{\text{worst}}$ can be thought of loosely as an estimate for the worst-case disturbance in the sense of the $\mathscr{H}_\infty$ norm. This illustrates the importance of an extra term as in (4.17). However, how to choose L_p to accomplish various objectives remains an open issue.

For the estimated-state feedback part of the covariance controller, we see that the feedback gain C_c given by (4.6) is identical to G in (4.16) if we replace $\bar{X}_c$ by X_p. This difference can be interpreted as the compensation for the estimation error due to noisy measurements by subtracting the error covariance $\bar{X}_p$ from X_p to obtain $\bar{X}_c$. Also note that the set of all assignable plant state covariances with full-order controllers is a subset of that with state feedback with the additional constraint $P < X_p$. This fact makes sense physically since the estimation error covariance P is zero if all the states are available without noise (P is the 'smallest' achievable estimation error covariance).

Finally, we see that the separation principle does not hold for the covariance controller, i.e., the state estimator and the estimated-state feedback gain cannot be designed separately since the determination of the estimator parameters A_c and B_c involves closed-loop information X_p, and the computation of the estimated-state feedback gain C_c requires the estimator information $\bar{X}_p$. Nevertheless, the plant state covariance X_p to be assigned can be specified without any knowledge of the estimator since the characterizations of X_p given by (4.1) and (4.2) involve only the *unique* solution P to (4.3). In other words, for any choice of assignable X_p (by (4.1), (4.2)), there always exists a full-order controller to assign X_p to the closed-loop system.

5. Conclusion

New assignability conditions are derived for linear time-invariant continuous-time systems with measurement noise. Inclusion of the measurement noise yields the standard Riccati equation as the second covariance assignability condition. Parametrizing the set of all stabilizing full-order controllers in terms of assignable covariances, the design of linear controllers reduces to a search for an assignable state covariance which carries desirable closed-loop properties.

Observer-based feedback structure of the full-order covariance controllers is interpreted in comparison with the previous results (with a state estimator with error covariance assignment and the state feedback covariance controller). It has been shown that the separation principle for the covariance controller does not hold, i.e., the state estimator cannot be designed independently of the estimated-state feedback gain. However, simultaneous assignment of the estimation error covariance and the plant state covariance can easily be done by solving a Riccati equation with a free matrix L_p of fixed dimension for the estimation error covariance, and by searching for an assignable plant state covariance in the convex set defined as the intersection of the linear affine set and the convex cone of positive-definite matrices. The state estimator part of the covariance controller can be specialized to the Kalman filter or the $\mathscr{H}_\infty$ central estimator by certain choices of the free matrix L_p.

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F. Minimal Energy Covariance Control

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Minimal energy covariance control

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In general, the covariance controller which assigns a specified state covariance X to the system is not unique, and the whole set of such controllers can be parametrized by a skew-symmetric matrix S of appropriate dimension. This paper describes how to minimize the control energy by using this freedom and reveals some properties of closed-loop system poles with respect to S.

1. Introduction

The main idea of the covariance control theory developed by Hotz and Skelton (1987) and Skelton and Ikeda (1989) is to choose a state covariance X according to different requirements on the system performance and robustness and then to design a controller so that the specified state covariance is assigned to the closed-loop system. For example, a performance requirement might be constraints on the output variances:

$$Y_{ii} \triangleq [CXC^*]_{ii} \leq \sigma_i, \quad i = 1, \ldots, n_y$$

where $[\cdot]_{ii}$ stands for the ith diagonal element of the matrix. Stability robustness might require constraints on the maximum singular value of X and Y (Corless *et al.* 1989). The computational errors in analogue controllers (from amplifier noise) can be minimized by placing constraints on the controller state covariance X_c:

$$[X_c]_{ii} = s, \; i = 1, \ldots, n_c$$

where X_c is the $(2, 2)$ block of

$$X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^* & X_c \end{bmatrix}$$

and s is a constant scaling factor (Hwang 1977, Liu, 1991).

In general, the controller which assigns the given X to the system is not unique, and the whole set of such assigning controllers can be parametrized by a skew-symmetric matrix $S = -S^*$ of appropriate dimension. Also, the system properties cannot be decided solely by the state covariance X. This paper describes how to use these free parameters to minimize the control energy and reveals some properties of the closed-loop system poles with respect to S.

The paper is outlined as follows. A brief review of the covariance assignment problem based on the result by Yasuda and Skelton (1990) is given in § 2. In § 3, the free parameter $S = -S^*$ for parametrizing the set of covariance controllers is used to minimize the mean squared control. Some properties of

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the closed-loop system poles with respect to the free parameter S, which can help to choose a specific S when considering also the location of the closed-loop system poles, are described in § 4. To illustrate the theory we give an example in § 5. Finally the paper is concluded with § 6 where the further research in this direction is discussed.

2. Parametrization of covariance controllers

Consider the following linear system of order n_x:

$$\left.\begin{aligned} \dot{x}_p &= A_p x_p + B_p u + D_p w \\ z &= M_p x_p \end{aligned}\right\} \tag{2.1}$$

where $x_p \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ the input, $w \in \mathbb{R}^{n_w}$ the plant disturbance, and $z \in \mathbb{R}^{n_z}$ the measurement. This system is driven by a linear controller of order n_c:

$$\left.\begin{aligned} \dot{x}_c &= A_c x_c + B_c z \\ u &= C_c x_c + D_c z \end{aligned}\right\} \tag{2.2}$$

By defining the matrices

$$\left.\begin{aligned} A &\triangleq \begin{bmatrix} A_p & 0 \\ 0 & 0 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} B_p & 0 \\ 0 & I_{n_c} \end{bmatrix}, \quad M \triangleq \begin{bmatrix} M_p & 0 \\ 0 & I_{n_c} \end{bmatrix} \\ G &\triangleq \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}, \quad D \triangleq \begin{bmatrix} D_p \\ 0 \end{bmatrix}, \quad x \triangleq \begin{bmatrix} x_p \\ x_c \end{bmatrix} \end{aligned}\right\} \tag{2.3}$$

where I_{n_c} denotes an $n_c \times n_c$ identity matrix, the closed-loop system can be written in the following form:

$$\dot{x} = (A + BGM)x + Dw \tag{2.4}$$

The state D covariance X of the system (2.4) (D for deterministic) (Yasuda and Skelton 1990) is defined by

$$\begin{aligned} X &\triangleq \sum_{i=1}^{r} \int_0^\infty x(i, t) x^*(i, t)\, dt, \; r = n_x + n_c + n_w \\ &= \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^* & X_c \end{bmatrix} \end{aligned}$$

where $x(i, t)$ denotes the state response of the system (2.4) when only the ith excitation is applied (from the total of $n_x + n_c + n_w$ excitations: $x_{p_\alpha}(0)$, $x_{c_\beta}(0)$, $w_\gamma(t) = w_\gamma \delta(t)$, $\alpha = 1, \ldots, n_x$, $\beta = 1, \ldots, n_c$, $\gamma = 1, \ldots, n_w$). It is known (Yasuda and Skelton 1990) that X is the unique solution of the Lyapunov equation

$$(A + BGM)X + X(A + BGM)^* + D_0 D_0^* = 0 \tag{2.5}$$

if the closed-loop system $A + BGM$ is stable, where

$$\begin{aligned} D_0 D_0^* &\triangleq DWD^* + X_0, \quad X_0 = \mathrm{diag}[\ldots x_{p_\alpha}^2(0) \ldots x_{c_\beta}^2(0) \ldots] \\ W &= \mathrm{diag}[\ldots w_\gamma^2 \ldots] \end{aligned}$$

Hence, the D covariance assignment problem is to find G such that (2.5) is satisfied for a specified $X > 0$. In this case, the stability of the closed-loop system is assured if $(A + BGM, D_0)$ is controllable, which is always true because D_0 is a non-singular matrix. The existence conditions for a controller of order n_c which will assign X to the closed-loop system are given as follows (Yasuda and Skelton 1990).

Theorem 2.1 (Yasuda and Skelton 1990): *A specified $X > 0$ can be assigned to the system by G if and only if*

$$(I - B_p B_p^+) Q_p (I - B_p B_p^+) = 0 \tag{2.6}$$

$$(I - M_p^+ M_p) \bar{Q}_p (I - M_p^+ M_p) = 0 \tag{2.7}$$

$$(I - LL^+)(I - M^+ M) X^{-1} Q B B^+ = 0 \tag{2.8}$$

are satisfied, where the superscript plus $(+)$ *denotes the Moore–Penrose inverse and*

$$\left.\begin{aligned}
Q_p &\triangleq X_p A_p^* + A_p X_p + D_{p0} D_{p0}^* \\
\bar{Q}_p &\triangleq \bar{X}_p^{-1}(\bar{X}_p A_p^* + A_p \bar{X}_p + D_{p0} D_{p0}^* + X_{pc} X_c^{-1} X_{0c} X_c^{-1} X_{pc}^*) \bar{X}_p^{-1} \\
\bar{X}_p &\triangleq X_p - X_{pc} X_c^{-1} X_{pc}^* \; (> 0) \\
L &\triangleq (I - M^+ M) X^{-1} B B^+ \\
Q &\triangleq X A^* + A X + D_0 D_0^* \\
D_0^* &\triangleq [D_{p0}^* D_{c0}^*]
\end{aligned}\right\} \tag{2.9}$$

For the characterization of all controllers which can assign X to the system we have the following theorem (Yasuda and Skelton 1990).

Theorem 2.2 (Yasuda and Skelton 1990): *Suppose that X is assignable; then all controllers which assign X as a state covariance to the system (2.4) are given as follows*:

$$G = G_1 + \bar{G}_2 \bar{S} \bar{G}_3 + Z - B^+ B Z M M^+ \tag{2.10}$$

where $\bar{S}$ is a skew-symmetric matrix and Z an arbitrary matrix of appropriate dimension respectively, and

$$G_1 \triangleq -\tfrac{1}{2} B^+ Q (2I - BB^+) X^{-1} M^+ + \tfrac{1}{2} B^+ (\Psi^* - \Psi) B B^+ X^{-1} M^+ \tag{2.11}$$

$$\bar{G}_2 \triangleq B^+ (I - L^+ L) \tag{2.12}$$

$$\bar{G}_3 \triangleq (I - L^+ L) B B^+ X^{-1} M^+ \tag{2.13}$$

with

$$\Psi \triangleq 2L^+(I - M^+ M) X^{-1} Q + [I - L^+(I - M^+ M) X^{-1}] Q L^+ L$$

Remark 2.1: By using the singular value decomposition of L given by

$$L = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Lambda_L & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{11}^* & V_{21}^* \\ V_{12}^* & V_{22}^* \end{bmatrix}$$

and defining

$$V_2 = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}$$

the dimension of the skew-symmetric matrix in (2.10) can be reduced, which means an elimination of the redundant 'free parameters'. In this case we have

$$G = G_1 + G_2SG_3 + Z - B^+BZMM^+ \tag{2.14}$$

where G_1 and Z are defined as before:

$$G_2 = B^+V_2^* \tag{2.15}$$

$$G_3 = V_2BB^+X^{-1}M^+ \tag{2.16}$$

and S is any skew-symmetric matrix of appropriate dimension which is obviously less than that of $\bar{S}$ in (2.10).

We shall assume that B_p has full column rank and M_p full row rank. In this case,

$$B^+BZMM^+ - Z \equiv 0 \tag{2.17}$$

and the set of all covariance controllers assigning X to the system is parametrized by $S = -S^*$. This freedom will be used in the following sections for improving some other system performances. □

3. Minimization of control energy

The cost function

$$v = \sum_{i=1}^{r}\int_0^\infty u^*(i, t)Ru(i, t)\,\mathrm{d}t \tag{3.1}$$

is often used as a measure for control energy in design. By using the relation

$$\begin{aligned} u &= C_cx_c + D_cM_px_p \\ &= [D_cM_p \quad C_c]\begin{bmatrix} x_p \\ x_c \end{bmatrix} \end{aligned} \tag{3.2}$$

the function (3.1) can be rewritten in the following way:

$$v = \mathrm{tr}(UR) \tag{3.3}$$

where

$$\begin{aligned} U &= \sum_{i=1}^{r}\int_0^\infty u(i, t)u^*(i, t)\,\mathrm{d}t \\ &= [D_cM_p \quad C_c]X[D_cM_p \quad C_c]^* \\ &= [D_cM_p \quad C_c]\begin{bmatrix} X_p & X_{pc} \\ X_{pc}^* & X_c \end{bmatrix}\begin{bmatrix} M_p^*D_c^* \\ C_c^* \end{bmatrix} \end{aligned}$$

with X being the state D covariance of the system. Hence,

$$v = \mathrm{tr}(D_cM_pX_pM_p^*D_c^*R + D_cM_pX_{pc}C_c^*R + C_cX_{pc}^*M_p^*D_c^*R + C_cX_cC_c^*R) \tag{3.4}$$

It is obvious that for the given M_p and R the value of v depends not only on the D covariance X but also on D_c and C_c which in turn are the functions of S for the given X as explained in § 2. In the rest of this section we shall describe how to minimize v with respect to S; that is, among all the controllers which assign a

specified X to the system, we shall determine that which consumes the least energy.

A basis for all skew-symmetric matrices $S = -S^* \in \mathbb{R}^{n_s \times n_s}$ can be chosen as follows:

$$S_{jk} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & 1_{jk} & & \vdots \\ 0 & -1_{kj} & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}, \text{ for } \begin{cases} j = 1, \ldots, n_s \\ k = 1, \ldots, n_s \text{ but } k \neq j \end{cases} \tag{3.5}$$

where 1_{jk} means that the (j, k) entry of the matrix is 1. Since the number of different S_{jk} is $m = \frac{1}{2}n_s(n_s - 1)$, we can rename the set $\{S_{jk}\}$ as $\{S_i\}$ with $i = 1, \ldots, m$. With this basis, the set of all skew-symmetric matrices can be parametrized with $\{\alpha_i, i = 1, \ldots, m\}$ as follows:

$$S = \alpha_1 S_1 + \cdots + \alpha_m S_m \tag{3.6}$$

Correspondingly, the covariance controller has the following form:

$$G = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} = G_1 + G_2 S G_3 = G_1 + \sum_{i=1}^{m} \alpha_i G_2 S_i G_3 \tag{3.7}$$

By blocking the matrices G_1 and $G_2 S_i G_3$ in the same way as that of blocking G

$$G_1 = \begin{bmatrix} D_{c0} & C_{c0} \\ B_{c0} & A_{c0} \end{bmatrix}, \quad G_2 S_i G_3 = \begin{bmatrix} D_{ci} & C_{ci} \\ B_{ci} & A_{ci} \end{bmatrix} \tag{3.8}$$

we have

$$C_c = C_{c0} + \sum_{i=1}^{m} \alpha_i C_{ci}, \quad D_c = D_{c0} + \sum_{i=1}^{m} \alpha_i D_{ci} \tag{3.9}$$

Substituting C_c and D_c in (3.4) with those in (3.9), we have

$$\begin{aligned} \operatorname{tr}(D_c M_p X_p M_p^* D_c^* R) &= \operatorname{tr}(D_{c0} M_p X_p M_p^* D_{c0}^* R) + 2\sum_{i=1}^{m} \alpha_i \operatorname{tr}(D_{c0} M_p X_p M_p^* D_{ci} R) \\ &\quad + \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i \alpha_j \operatorname{tr}(D_{ci} M_p X_p M_p^* D_{cj}^* R) \\ &= c_1 + 2\alpha^{\mathrm{T}} b_1 + \alpha^{\mathrm{T}} Q_1 \alpha \end{aligned} \tag{3.10}$$

where

$$\alpha^{\mathrm{T}} = [\alpha_1, \ldots, \alpha_m] \tag{3.11}$$

$$c_1 = \operatorname{tr}(D_{c0} M_p X_p M_p^* D_{c0}^* R) \in \mathbb{R}^{1\times 1} \tag{3.12}$$

$$b_1 = \begin{bmatrix} \operatorname{tr}(D_{c0} M_p X_p M_p^* D_{c1}^* R) \\ \vdots \\ \operatorname{tr}(D_{c0} M_p X_p M_p^* D_{cm}^* R) \end{bmatrix} \in \mathbb{R}^{m\times 1} \tag{3.13}$$

$$\begin{aligned} Q_1 &= \begin{bmatrix} \operatorname{tr}(D_{c1} M_p X_p M_p^* D_{c1}^* R) & \cdots & \operatorname{tr}(D_{c1} M_p X_p M_p^* D_{cm}^* R) \\ \vdots & & \vdots \\ \operatorname{tr}(D_{cm} M_p X_p M_p^* D_{c1}' R) & \cdots & \operatorname{tr}(D_{cm} M_p X_p M_p^* D_{cm}^* R) \end{bmatrix} \\ &= Q_1^* \in \mathbb{R}^{m\times m} \end{aligned} \tag{3.14}$$

$$\operatorname{tr}(D_c M_p X_{pc} C_c^* R) = \operatorname{tr}(D_{c0} M_p X_{pc} C_{c0} R) + \sum_{i=1}^{m} \alpha_i \operatorname{tr}(D_{c0} M_p X_{pc} C_{ci}^* R)$$

$$+ \sum_{i=1}^{m} \alpha_i \operatorname{tr}(D_{ci} M_p X_{pc} C_{c0}^* R) + \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i \alpha_j \operatorname{tr}(D_{ci} M_p X_{pc} C_{cj}^* R)$$

$$= c_2 - \alpha^* b_2 + \alpha^* Q_2 \alpha \tag{3.15}$$

where

$$c_2 = \operatorname{tr}(D_{c0} M_p X_{pc} C_{c0}^* R) \tag{3.16}$$

$$b_2 = \begin{bmatrix} \operatorname{tr}(D_{c0} M_p X_{pc} C_{c1}^* R) + \operatorname{tr}(D_{c1} M_p X_{pc} C_{c0}^* R) \\ \vdots \\ \operatorname{tr}(D_{c0} M_p X_{pc} C_{cm}^* R) + \operatorname{tr}(D_{cm} M_p X_{pc} C_{c0}^* R) \end{bmatrix} \tag{3.17}$$

$$Q_2 = \begin{bmatrix} \operatorname{tr}(D_{c1} M_p X_{pc} C_{c1}^* R) & \dots & \operatorname{tr}(D_{c1} M_p X_{pc} C_{cm}^* R) \\ \vdots & & \vdots \\ \operatorname{tr}(D_{cm} M_p X_{pc} C_{c1}^* R) & \dots & \operatorname{tr}(D_{cm} M_p X_{pc} C_{cm}^* R) \end{bmatrix} \tag{3.18}$$

$$\operatorname{tr}(C_c X_{pc}^* M_p^* D_c^* R) = c_3 + \alpha^* b_3 + \alpha^* Q_3 \alpha \tag{3.19}$$

where

$$c_3 = c_2, \quad b_3 = b_2, \quad Q_3 = Q_2^* \tag{3.20}$$

$$\operatorname{tr}(C_c X_c C_c^* R) = \operatorname{tr}(C_{c0} X_c C_{c0}^* R) + 2\sum_{i=1}^{m} \alpha_i \operatorname{tr}(C_{c0} X_c C_{ci}^* R)$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i \alpha_j \operatorname{tr}(C_{ci} X_c C_{cj}^* R)$$

$$= c_4 + 2\alpha^* b_4 + \alpha^* Q_4 \alpha \tag{3.21}$$

where

$$c_4 = \operatorname{tr}(C_{c0} X_c C_{c0}^* R)$$

$$b_4 = \begin{bmatrix} \operatorname{tr}(C_{c0} X_c C_{c1}^* R) \\ \vdots \\ \operatorname{tr}(C_{c0} X_c C_{cm}^* R) \end{bmatrix}$$

$$Q_4 = \begin{bmatrix} \operatorname{tr}(C_{c1} X_c C_{c1}^* R) & \dots & \operatorname{tr}(C_{c1} X_c C_{cm}^* R) \\ \vdots & & \vdots \\ \operatorname{tr}(C_{cm} X_c C_{c1}^* R) & \dots & \operatorname{tr}(C_{cm} X_c C_{cm}^* R) \end{bmatrix}$$

Hence, we have

$$\begin{aligned} v &= \sum_{i=1}^{r} \int_0^\infty u^*(i, t) R u(i, t)\, \mathrm{d}t \\ &= c + 2\alpha^* b + \alpha^* Q \alpha \end{aligned} \tag{3.22}$$

where

$$\left.\begin{aligned} c &= c_1 + c_2 + c_3 + c_4 \\ b &= b_1 + \tfrac{1}{2} b_2 + \tfrac{1}{2} b_3 + b_4 \\ Q &= Q_1 + Q_2 + Q_3 + Q_4 \end{aligned}\right\} \tag{3.23}$$

Since $v \geqslant 0$ is quadratic in α we have $Q = Q^* \geqslant 0$. Further we can draw the following conclusion.

Theorem 3.1: *For a specified state covariance X, the cost function*

$$v = \sum_{i=1}^{r} \int_0^{\infty} u^*(i, t) R u(i, t)\, \mathrm{d}t$$

$$= c + 2\alpha b + \alpha^* Q \alpha$$

where c, b and Q are previously defined by (3.23), *has a global minimum with respect to α, if and only if $(I - QQ^+)b = 0$, and the minimum can be achieved by any α satisfying*

$$Q\alpha + b = 0 \tag{3.24}$$

Proof: The proof follows from $\partial v / \partial \alpha = 2b + (Q + Q^*)\alpha$, $\partial^2 v / \partial \alpha^2 = Q + Q^* \geqslant 0$, and the existence condition for a solution to the linear algebra problem (3.24) (Strang 1988). □

4. Some properties of the poles of covariance control systems

As described in § 2, the set of controllers which assign a specified state covariance X to the system can be parametrized by $S = -S^*$, and hence the location of closed-loop system poles is also dependent only on S for the given X. Some properties of the closed-loop poles resulting from this dependence will be studied in the remainder of this section.

Theorem 4.1: *Suppose that a specified state covariance X can be assigned to the system. Among all controllers that assign X, the sum of the closed loop poles is constant.*

For the proof of Theorem 4.1 we need the following lemma.

Lemma 4.2: *For any skew-symmetric matrix $S = -S^*$ and any symmetric matrix $X = X^*$, we have*

$$\operatorname{tr}(SX) = 0 \tag{4.1}$$

Proof:

$$\operatorname{tr}(SX) = \operatorname{tr}(XS)$$

$$\operatorname{tr}(SX) = \operatorname{tr}[(SX)^*] = -\operatorname{tr}(XS)$$

that is

$$\operatorname{tr}(XS) = -\operatorname{tr}(XS) = 0$$ □

By using Lemma 4.2 we can prove Theorem 4.1 in the following way.

Proof of Theorem 4.1:

$$\sum_{i=1}^{n_x + n_c} \lambda_i (A + BGM) = \sum_{i=1}^{n_x + n_c} \lambda_i (A + BG_1 M + BG_2 S G_3 M)$$

$$= \operatorname{tr}(A + BG_1 M + BG_2 S G_3 M)$$

$$= \operatorname{tr}(A + BG_1 M) + \operatorname{tr}(BG_2 S G_3 M)$$

From (2.5), (2.14) and (2.17), we have

$$[A + B(G_1 + G_2SG_3)M]X + X[A + B(G_1 + G_2SG_3)M]^* + D_0D_0^* = 0 \tag{4.2}$$

Since this is true for all $S = -S^*$, we can set $S = 0$ and obtain

$$(A + BG_1M)X + X(A + BG_1M)^* + D_0D_0^* = 0 \tag{4.3}$$

By subtracting (4.3) from (4.2), we have

$$(BG_2SG_3M)X + X(BG_2SG_3M)^* = 0 \tag{4.4}$$

which means that

$$BG_2SG_3MX = \hat{S} \tag{4.5}$$

where $\hat{S}$ is a skew-symmetric matrix. Equation (4.5) can also be written as follows:

$$BG_2SG_3M = \hat{S}X^{-1} \tag{4.6}$$

Now by using Lemma 2, we obtain

$$\mathrm{tr}(BG_2SG_3M) = \mathrm{tr}(\hat{S}X^{-1}) = 0 \tag{4.7}$$

and hence

$$\sum_{i=1}^{n_x+n_c} \lambda_i(A + BGM) = \mathrm{tr}(A + BG_1M)$$

which is independent of S. This completes the proof. □

Remark 4.1: Since the system $[A + B(G_1 + G_2SG_3)M,\ D_0]$ is always controllable and X is chosen to be positive definite, the closed-loop system $A + B(G_1 + G_2SG_3)M$ retains its stability for any skew-symmetric matrix S. Theorem 4.1 tells us in addition that the real part of every closed-loop pole is bounded by some number which is independent of S.

In the following we study another property of the closed-loop system poles for the case where $\dim(S) = 2[\dim(S) \leq \mathrm{rank}\, B \leq$ number of control inputs]. In this case S can be parametrized by a single parameter α in the following way:

$$S = \alpha S_2, \quad \forall \alpha \tag{4.8}$$

where

$$S_2 \triangleq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the characteristic polynomial of the system is then

$$\begin{aligned} \Delta(s) &\triangleq |sI - (A + BG_1M + \alpha BG_2S_2G_3M)| \\ &= |sI - (A + BG_1M)|\,|I - \alpha(sI - A - BG_1M)^{-1}BG_2S_2G_3M| \end{aligned} \tag{4.9}$$

In general, we almost always have (Davison and Wang 1973)

$$\lambda_i(A + BG_1M) \neq \lambda_j(A + BG_1M + \alpha BG_2S_2G_3M),\ \forall i,\ j,\ \alpha \neq 0$$

if $(A + BG_1M,\ B)$ is controllable. This condition is satisfied because we assume

that (A, B) is controllable. Hence, we have

$$r_i[\Delta(s)] = r_i[|I - \alpha(sI - A - BG_1M)^{-1}BG_2S_2G_3M|], \quad \forall\alpha \neq 0 \qquad (4.10)$$

where $r_i(\cdot)$ denotes the ith root of a polynomial. Again by exploiting a well known result in the theory of multivariable root loci (Kouvaritakis and Shaked 1976), we obtain

$$\lim_{\alpha \to \infty} \{r_i[\Delta(s)]\} = z_i[(sI - A - BG_1M)^{-1}BG_2S_2G_3M] \qquad (4.11)$$

with $z_i(\cdot)$ denoting the ith transmission zero of a system which can be either finite or infinite. Since the system in (4.11) is strictly proper, we can draw the following conclusion by considering Theorem 4.1.

Corollary 4.3: *For the case where* $\dim(S) = 2$, *there is at least a pair of poles of the covariance control system tending to infinity in the direction parallel to* $+j\omega$ *axis and* $-j\omega$ *axis respectively as* α *tends to infinity.*

Remark 4.2: Theorem 4.1 and Corollary 4.3 describe an interesting property of covariance control system poles. This property can be used to study the root loci of covariance control systems, for example to locate the system poles in a certain region in the complex place. This property will be shown in Example 5.1. □

5. Example

To illustrate the theory presented in previous sections, we give an example here.

Example 5.1: By solving the problem of assigning

$$X_p = 10^{-4}\begin{bmatrix} 8 & -5 & 0 & 0 \\ -5 & 5 & 0 & 0 \\ 0 & 0 & 4\cdot5 & -5 \\ 0 & 0 & -5 & 7\cdot5 \end{bmatrix}$$

to the plant (a lightly damped flexible beam with two modes (Hsieh 1990)),

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0\cdot1 & -0\cdot01 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0\cdot01 & -0\cdot005 \end{bmatrix}$$

$$B_p = \begin{bmatrix} 0 & 0 \\ 2 & 1 \\ 0 & 0 \\ 3 & 5 \end{bmatrix}, \quad D_pWD_p^* = 10^{-3}I, \quad X_0 = 0$$

via state feedback, we have from (2.11)–(2.13)

$$G_s = -\tfrac{1}{2}B_p^+Q_p(2I - B_pB_p^+)X_p^{-1} + B_p^+\bar{S}B_pB_p^+X_p^{-1}$$

Using the singular value decomposition of B_p given by

$$B_p = [U_{b1} \quad U_{b2}]\begin{bmatrix} \Lambda_b \\ 0 \end{bmatrix} V_b$$

we obtain

$$G_s = -\tfrac{1}{2}B_p^- Q_p(2I - B_p B_p^-)X_p^{-1} + B_p^+ U_{b1} S U_{b1}^* X_p^{-1}$$

By minimizing the cost function

$$\upsilon = \mathrm{tr}(UR) = \mathrm{tr}(G_s X_p G_s^* R)$$

where R is chosen as

$$R = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

we obtain the optimal S:

$$S_{\mathrm{opt}} = \alpha_{\mathrm{opt}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

where $\alpha_{\mathrm{opt}} = 1{\cdot}8847 \times 10^{-4}$. and the corresponding minimal cost value

$$\upsilon_{\min} = 3{\cdot}4258 \times 10^{-3} \quad \text{for } S = \alpha_{\mathrm{opt}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which is smaller than either

$$\upsilon = 3{\cdot}5133 \times 10^{-3} \quad \text{for } S = 0 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

or

$$\upsilon = 6{\cdot}1567 \times 10^{2} \quad \text{for } S = S_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

If our purpose is not to minimize the control energy but to assure that all the poles of the covariance control system be located within the sector of $\pm 45°$ in the complex plane, we can find from the root loci in Fig. 5.1 the critical $\alpha_c = 8{\cdot}35 \times 10^{-4}$ such that $\alpha < \alpha_c$ guarantees that the damping ratio of the covariance control system is larger than 0·707. The root loci of the system with respect to $S = \alpha S_2$ are given in Fig. 5.1.

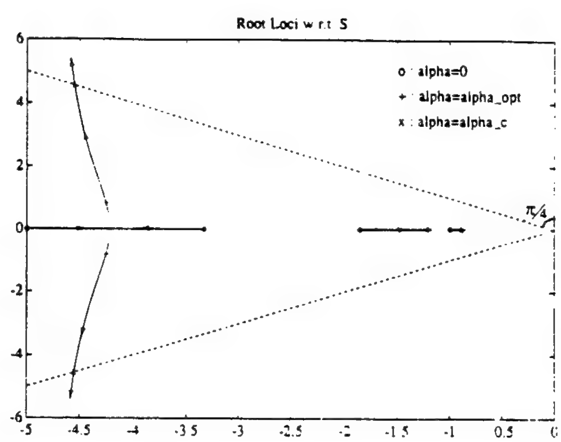

Figure 5.1. Root focus for constant covariance.

To study the time response of the covariance control system, three different S ($S = \alpha_{opt}S_2$, $\alpha_c S_2$ and S_2) are chosen, and the impulse responses of the corresponding closed-loop systems are shown in Fig. 5.2. It can be seen from Fig. 5.2 that these three different systems have very similar impulse responses $y(t)$. This is because these systems have the same output covariance matrix Y (recall from the theory that S does not influence Y or X). Notice, however, the influence of S on the control variable from Fig. 5.2.

Remark 5.1: Some poles of the covariance control system with $S = S_2$ are too large to be shown in the properly scaled Fig. 5.1, and hence the poles for this case have not been plotted here. To conserve space, only $y(1)$ and $u(1)$ are plotted to represent the output $y(t)$ and control variable $u(t)$.

6. Conclusion

It is well known (Skelton and Ikeda 1989, Yasuda and Skelton 1990) that all controllers which assign a specified state covariance X to the system can be parametrized by a free skew-symmetric matrix S. In this paper, this freedom is

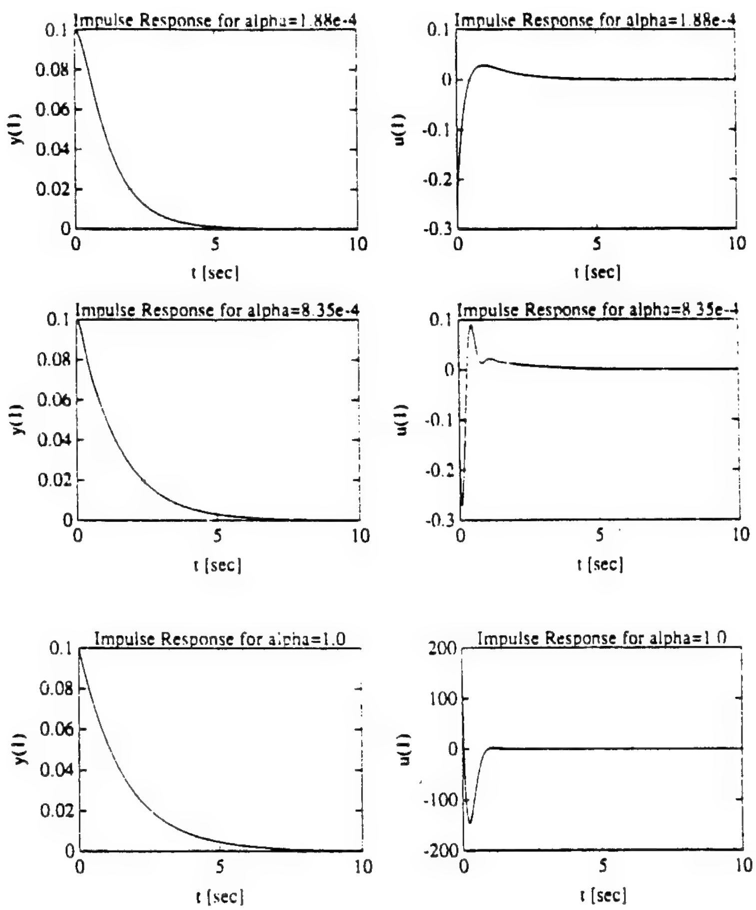

Figure 5.2. The impulse responses of the closed-looped systems corresponding to Fig. 1.

used to minimize the control energy; that is, among all the controllers which assign a given X to the system we have derived the controller which uses the least energy.

Some properties of closed-loop system poles with respect to S are also revealed. In classical root locus theory (for excess poles over zero of 2 or more), it is known that, as the open loop gain goes to infinity, *the sum of the closed-loop poles is a constant* (but stability is not guaranteed). A modern-day counterpart of this 'pole conservation' principle is presented by showing that, as the controller is changed over the entire set of controllers that preserve the same positive state covariance matrix, *the sum of the closed-loop poles is a constant (and stability is guaranteed)*.

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G. Parametrization of All Linear Compensators for Discrete-time Stochastic Parameter Systems

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Parametrization of All Linear Compensators for Discrete-time Stochastic Parameter Systems*

ENGIN YAZ† and ROBERT E. SKELTON‡

A characterization of all assignable state covariances and a parametrization of all linear stabilizing output feedback controllers that achieve this assignment are given for discrete-time stochastic parameter systems.

Key Words—Stochastic control; stability robustness; discrete-time systems; feedback control.

Abstract—For discrete-time stochastic parameter systems, this paper presents a characterization of all state covariances assignable by a linear controller and a parametrization of all controllers that achieves a desired covariance. These results indirectly provide the parametrization of all linear fixed-order compensators which are mean square stabilizing for this class of systems. The paper also includes robustification of the derived compensators and an example to illustrate the results.

INTRODUCTION

THIS PAPER considers the static and dynamic output feedback control of discrete-time stochastic parameter systems. Such systems are also called state (control) dependent or multiplicative noise models or systems with white parameters. The interest in such models stems from several application areas such as satellite attitude control (McLane, 1971; Sagirow, 1972), vibration study of structures and mechanical systems (Bolatin, 1984; Ibrahim, 1985), chemical reactor control (Wagenaar and DeKoning, 1989), macro-economics (Aoki, 1976), population dynamics (Bartlett, 1960; Tsokos and Padgett, 1974; Mohler and Kolodziej, 1980), random amplitude modulation in filtering (Nahi, 1969; Rajasekaran *et al.*, 1971; Tugnait, 1981), random roundoff errors in digital control (Liu and Kaneko, 1969; Rink and Chong, 1979a, b; VanWingerden and DeKoning, 1984), analysis of circuits (Willsky and Marcus, 1976; Michel and Miller, 1977), and recently in robustness studies and robustifying existing controllers (Willems and Willems, 1983; Yaz and Yildizbayrak, 1985; Bernstein and Haddad, 1987b; Yaz, 1988a, 1989b, 1992).

We will confine our treatment to discrete-time system models. Such models, for example, may be the result of random sampling of continuous-time processes where the randomness may be caused by the choice of a human operator in the control loop, a mere absence of data, or it may be carried out to observe the underlying inter-sample behavior (DeKoning, 1988). Digital control using a computer with finite word length (Liu and Kaneko, 1969; Rink and Chong, 1979a, b; VanWingerden and DeKoning, 1984) is another reason for the appearance of discrete-time stochastic parameter models. When continuous-time systems given by an Ito differential equation with state- and control-multiplied Wiener noise effect are uniformly sampled, such models arise again. More recently, random parameters are introduced into the problem formulation to achieve an extra degree of stability-robustness in controller and estimator designs (Willems and Willems, 1983; Yaz and Yildizbayrak, 1985; Bernstein, 1987a, b; Yaz, 1988a, 1988b, 1992) which again necessitates the consideration of discrete-time stochastic parameter models.

Motivated by the application possibilities enumerated above and also by the elegance of the mathematics involved, many researchers contributed to this field. We will simply give a few references and refer the reader to the excellent survey papers (Malyshev and Pakshin, 1990a, b). The dynamic output feedback control of discrete-time stochastic parameter systems has

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been considered in Pakshin (1978), Bernstein and Haddad (1987b), Yaz (1988b), Phillis (1990), DeKoning (1992). Pakshin (1978) is on the necessary conditions for the mean square optimality of a strictly proper full-order (has the same dimension as the state) linear compensator for such systems. Bernstein and Haddad (1987b) obtain necessary optimality conditions for both static and fixed-order dynamic output feedback controllers. Yaz (1988b) derives sufficient conditions for the mean-square stabilization (the results are also true for almost sure stabilization) by strictly proper full-order compensators. Phillis (1990) operates in the framework of Pakshin (1978) but extends the results to minimax control for systems with unknown noise covariances. DeKoning (1992) introduces the concept of mean-square compensatability which is the ability to mean-square stabilize such a system by a linear strictly proper time-invariant compensator of full-order. Necessary and sufficient conditions for the compensatability as well as the computation of optimal compensators are presented.

In the present work, we will be interested in static and fixed-order dynamic output feedback covariance assignment for discrete-time stochastic parameter systems. First, necessary and sufficient conditions for the assignability by a static output feedback of a given covariance (the steady state second moment of the system state) will be derived. Next, all controllers that will achieve this assignment will be parametrized. The assignability conditions will be interpreted in light of existing system theoretic concepts for this class of systems. Then, this design is robustified to allow parameter variations in the operation and a robust controller which maintains mean-square and almost-sure boundedness of the system state is given. All of these results are then generalized to the case of fixed-order dynamic compensation. An example is given to illustrate the use of the proposed techniques. This paper is a much improved version of its predecessor (Skelton *et al.*, 1991) in several ways: first, the present model additionally includes control-dependent noise in the state equation and state-dependent noise and additive noise in the measurement equation in a unified framework, where the multiplicative noises are correlated (this helps in realistic modeling of several physical phenomena). Second, a different and more direct approach is used in deriving the results with interpretations of assignability conditions using well-known system theoretic concepts. Third, the robustness study is more general in that it encompasses both deterministic and stochastic model uncertainties.

MODEL AND PRELIMINARIES

Consider the discrete-time stochastic parameter system

$$x^s_{k+1} = A^s_k(\omega)x^s_k + B^s_k(\omega)u_k + v_k(\omega), \qquad (1)$$

where $x^s_k \in R^n$, $u_k \in R^m$ and A^s_k, B^s_k, and v_k are all composed of white noise (time-wise independent) elements. A^s_k and B^s_k are uncorrelated with v_k which has a zero mean (denoted by $\bar{v}_k = E\{v_k\} = 0$) and a constant covariance of Σ_v. $\omega \in \Omega$ where $(\Omega, \mathcal{H}, \mathcal{P})$ is a fixed probability space. The measurement equation is

$$y_k = C^s_k(\omega)x^s_k + w_k(\omega), \qquad (2)$$

where $y_k \in R^p$, $p \le n$ and the matrix of white noise elements C^s_k is uncorrelated with the additive noise vectors v_k and w_k with zero mean and covariance $\Sigma_w > 0$, but we allow correlation of A^s_k, B^s_k, and C^s_k. For simplicity, let w_k be uncorrelated with all other noises, let all multiplicative noises be weakly stationary and all noises be independent of the initial state x_0. Note that the above representation includes deterministic parameter systems (by simply setting the covariances of multiplicative noises to zero) and the usual, more structured parameter representation with $A_k(\omega) = A_0 + \sum_{i=1}^{r} a^i_k(\omega)A^i_1$, etc, although the input matrix $B^s_k(\omega)$ in our model will be restricted later to obtain stronger results.

We include a few system theoretical notions for the above class of systems to facilitate the discussion. The system (1) or the pair (A^s_k, B^s_k) is called mean square (m.s.) stabilizable if there exists a constant state feedback controller such that this controller renders the state of system (1) m.s. bounded ($\sup_{k\ge 0} E\{\|x_k\|^2\} < \infty$) for all Σ_v. A simple test for m.s. stabilizability is given in DeKoning (1982). It turns out that a m.s. stabilizing control is almost surely (a.s.), or with probability one, stabilizing for this class of systems (Yaz, 1988a) ($\sup_{k\ge 0} \|x_k\| < \infty$ with probability one). The (A^s_k, C^s_k) pair for the model $x^s_{k+1} = A^s_k(\omega)x^s_k$, $y_k = C^s_k(\omega)x^s_k$ is called m.s. observable if $E\{\|y_k\|^2\} = 0$ for all $k \ge 0$ implies $x_0 = 0$. The definition is shown in DeKoning (1982) to result in the following test for m.s. observability. Define $\mathcal{A}: R^{n\times n} \to R^{n\times n}$ by

$$\mathcal{A}(X) \triangleq E\{A^{s^T}XA^s\}. \qquad (3)$$

Note that the subscript k is dropped due to the weak stationarity of the noise elements. The necessary and sufficient condition for the m.s. observability is that the observability Grammian

satisfies

$$\sum_{k=0}^{\ell} \mathscr{A}^k(\overline{C^{s^T}C^s}) > 0, \quad \ell = [n(n+1)/2] - 1, \quad (4)$$

where $\mathscr{A}^k = \mathscr{A}(\mathscr{A}^{k-1})$ and $\mathscr{A}^0(X) = X$. Obviously for a $Q > 0$, $(A_k^s, Q^{1/2})$ is always m.s. observable for any A_k^s.

The following results will also be useful in the sequel.

Lemma 1 (*DeKoning*, 1982). Consider the stochastic algebraic Lyapunov equation:

$$X = \mathscr{A}(X) + \Sigma_v, \quad \Sigma_v = \Sigma_v^T \geq 0,$$

where $\mathscr{A}$ is given by (3).

(i) If $\mathscr{A}$ is a stable transformation [which is equivalent to the m.s. boundedness of the state of unforced system (1)], then there exists an $X \geq 0$ which is the unique real symmetric solution.

(ii) If there exists a solution $X \geq 0$, and $(A_k^s, \Sigma_v^{1/2})$ is m.s. observable, then $\mathscr{A}$ is a stable transformation and $X > 0$.

Lemma 2. The stability of the transformation $\mathscr{A}$ given by (3) is equivalent to that of $\mathscr{A}^*$: $R^{n\times n} \to R^{n \times n}$ given by

$$\mathscr{A}^*(X) = E\{A^s X A^{s^T}\}.$$

Proof. The proof simply follows from the necessary and sufficient condition for the m.s. stability of $\mathscr{A}$ (Kalman, 1961):

$$\rho(\mathscr{A}) = \rho(\overline{A^s \otimes A^s}) < 1,$$

where the first one is the operator's spectral radius, and the second one is the spectral radius of the matrix which is the expected value of the Kronecker square of the A_k^s matrix. If the properties $(A \otimes B)^T = A^T \otimes B^T$, $\rho(A) = \rho(A^T)$, $E\{A^T(\omega)\} = [E\{A(\omega)\}]^T$ are used in the above condition, we obtain

$$\rho(\mathscr{A}) = \rho(\mathscr{A}^*) < 1,$$

completing the proof.

STATIC OUTPUT FEEDBACK

(i) *Assignability and covariance assignment*

Let us now assume that $B_k^s(\omega) = b_k(\omega)B^s$ where $b_k(\omega)$ is a scalar white noise sequence, and try to assign the state covariance (second moment) $X \triangleq \lim_{k\to\infty} E\{x_k x_k^T\}$ by the control

$$u_k = Ky_k. \quad (5)$$

The closed-loop system is

$$x_{k+1}^s = (A_k^s + b_k B^s K C_k^s)x_k^s + v_k + b_k B^s K w_k \quad (6)$$

and if the control is mean square stabilizing, the steady state covariance of the state vector satisfies

$$X = \overline{(A^s + bB^sKC^s)X(A^s + bB^sKC^s)^T} + \Sigma_v + \overline{b^2}B^sK\Sigma_wK^TB^{s^T}. \quad (7)$$

Independence of the noises is utilized in the simplification resulting in (7). Upon rearranging, (7) yields

$$X - \overline{A^sXA^{s^T}} - \Sigma_v = B^sK\overline{bC^sXA^{s^T}} + \overline{bA^sXC^{s^T}}K^TB^{s^T} + B^sK[\overline{b^2C^sXC^{s^T}} + \overline{b^2}\Sigma_w]K^TB^{s^T}. \quad (8)$$

Completing the square on the right side, we will obtain

$$X - \overline{A^sXA^{s^T}} - \Sigma_v = (B^sK + G(X))[\overline{b^2C^sXC^{s^T}} + \overline{b^2}\Sigma_w] \times (B^sK + G(X))^T - G(X)[\overline{b^2C^sXC^{s^T}} + \overline{b^2}\Sigma_w]G^T(X), \quad (9)$$

where

$$G(X) \triangleq \overline{bA^sXC^{s^T}}(\overline{b^2C^sXC^{s^T}} + \overline{b^2}\Sigma_w)^{-1}. \quad (10)$$

Rearranging once more yields

$$X - \overline{A^sXA^{s^T}} - \Sigma_v + \overline{bA^sXC^{s^T}} \times (\overline{b^2C^sXC^{s^T}} + \overline{b^2}\Sigma_w)^{-1}\overline{bC^sXA^{s^T}} = (B^sK + G(X))TT^T(B^sK + G(X))^T. \quad (11)$$

At this point, X must be chosen such that it renders the left side of (11) non-negative definite with rank not exceeding p, which is required by the form of the right side of (11). In that case, we have

$$LL^T = (B^sK + G(X))TT^T(B^sK + G(X))^T \quad (12)$$

with the obvious definition of the new variables $L \in R^{n\times p}$, $T \in R^{p\times p}$ with $\det[T] \neq 0$. Hence, for an orthogonal matrix $U \in R^{p\times p}$, it is true that

$$LU = (B^sK + G(X))T \quad (13)$$

or

$$B^sK = LUT^{-1} - G(X) \quad (14)$$

which can be solved for K (Collins and Skelton, 1987) as

$$K = B^{s^{\dagger}}(LUT^{-1} - G(X)) + (I_p - B^{s^{\dagger}}B^s)Z, \quad (15)$$

where $B^{s^{\dagger}}$ denotes the unique Moore–Penrose pseudo-inverse, for any arbitrary Z if and only if

$$B^sB^{s^{\dagger}}[LUT^{-1} - G(X)] = LUT^{-1} - G(X) \quad (16)$$

or

$$(I - B^sB^{s^{\dagger}})LU = (I - B^sB^{s^{\dagger}})G(X)T,$$

which is equivalent to

$$(I - B^sB^{s\dagger})[X - \overline{A^sXA^{s^T}} - \Sigma_v](I - B^sB^{s\dagger}) = 0 \quad (17)$$

using (10) and (12). The matrix U can also be found by (Collins and Skelton, 1987)

$$U = \mathcal{V}_1 \text{ block diag } [I_p, \bar{U}]\mathcal{V}_2^T, \quad (18)$$

where the singular value decompositions $(I - B^sB^{s\dagger})L = \mathcal{W}_1 \text{ block diag } [\Lambda, 0]\mathcal{V}_1^T$, $(I - B^sB^{s\dagger})G(X)T = \mathcal{W}_1 \text{ block diag } [\Lambda, 0]\mathcal{V}_2^T$ are used with $\bar{U} \in R^{(p-r_u)\times(p-r_u)}$ being row-orthonormal, and $\Lambda \in R^{p\times p}$ contains all the nonzero singular values. Summing up, we have

Theorem 1. A given state covariance X is assignable to system (1) by some control (5) if and only if it makes the left side of (11) non-negative definite with rank not exceeding p and it satisfies the consistency condition (17). If X is assignable, all controllers assigning X are given by (15), (18).

Note that if the covariance assignment is successful and $(A_k^{s^T}, \Sigma_v^{1/2})$ is m.s. observable, then by Theorem 4.5 of DeKoning (1982), $([A_k^s + b_kB^sKC_k^s]^T, [\Sigma_v^{1/2}, (\overline{b^2})^{1/2}B^sK\Sigma_w^{1/2}])$ is m.s. observable, and Lemma 1, combined with the results of Yaz (1988a) guarantees a.s. boundedness of the system state.

(ii) *On the existence of solutions to assignability conditions*

Now let us interpret the assignability conditions starting with

$$X - \overline{A^sXA^{s^T}} - \Sigma_v + \overline{bA^sXC^{s^T}} \times (\overline{b^2C^sXC^{s^T}} + \overline{b^2}\Sigma_w)^{-1}\overline{bC^sXA^{s^T}} \geq 0. \quad (19)$$

It is easy to see that this inequality can be rearranged to give

$$X \geq \overline{[A^{s^T} - bC^{s^T}G^T(X)]^T X [A^{s^T} - bC^{s^T}G^T(X)]} + \overline{b^2}G(X)\Sigma_wG^T(X) + \Sigma_v \triangleq \Phi(X), \quad (20)$$

where $G(X)$ is given by (10). It is shown in Yaz (1989a) that the existence of X satisfying (20) is equivalent to the m.s. stabilizability of $(A_k^{s^T}, b_kC_k^{s^T})$ for a m.s. observable pair $(A_k^{s^T}, \Sigma_v^{1/2})$.

Now, considering equation (17), we can write

$$(I - B^sB^{s\dagger}) \times [X - \overline{(A^s + bB^sK)X(A^s + bB^sK)^T} - \Sigma_v] \times (I - B^sB^{s\dagger}) = 0, \quad (21)$$

by using the properties of the Moore–Penrose pseudo inverse:

$$(I - B^sB^{s\dagger})B^s = 0, \quad B^{s^T}(I - B^sB^{s\dagger}) = 0 \quad (22)$$

for an arbitrary K. So, according to Lemma 1, if (A_k^s, b_kB^s) pair is m.s. stabilizable (i.e. there exists a K such that $(A_k^s + b_kB^sK)$ leads to a stable transformation, then there exists a solution $X \geq 0$ to the equation

$$X = \overline{(A^s + bB^sK)X(A^s + bB^sK)^T} + \Sigma_v, \quad (23)$$

which implies (21). We have actually used the property presented in Lemma 2 that the transformations $\mathcal{A}$ and $\mathcal{A}^*$ have the same stability properties. Therefore, one can see that the m.s. stabilizability of (A_k^s, b_kB^s) is sufficient for the existence of a solution to the consistency equation (17).

The above discussion is included to relate the assignability conditions individually to well-known system theoretic concepts. Obviously, for assignability, the same covariance must satisfy all the assignability conditions simultaneously which means that the system theoretic conditions associated with individual assignability conditions as a whole are neither necessary nor sufficient for the assignability of a specific covariance.

(iii) *Robust design*

It is possible to robustify this controller so as to accommodate erroneous modeling of parameter statistics or parameter perturbations by making the closed-loop system a prescribed degree of m.s. stable via modifying the design equations. Let us change equation (7) to

$$X_\alpha = \alpha^2\overline{(A^s + bB^sK_\alpha C^s)X_\alpha(A^s + bB^sK_\alpha C^s)^T} + \alpha^2\overline{b^2}B^sK_\alpha\Sigma_wK_\alpha^TB^{s^T} + \Sigma_v \quad (24)$$

with $\alpha > 1$ which will modify the non-negativity condition as

$$X \geq \alpha^2[\overline{A^sXA^{s^T}} - \overline{bA^sXC^{s^T}} \times (\overline{b^2C^sXC^{s^T}} + \overline{b^2}\Sigma_w)^{-1}\overline{bC^sXA^{s^T}}] + \Sigma_v \triangleq \Phi(\alpha, X_\alpha) \quad (25)$$

and the consistency condition (18) changes to

$$(I - B^sB^{s\dagger})[X_\alpha - \alpha^2\overline{A^sX_\alpha A^{s^T}} - \Sigma_v] \times (I - B^sB^{s\dagger}) = 0, \quad (26)$$

which if holds yields the control gain:

$$K_\alpha = B^{s\dagger}[L_\alpha UT_\alpha^{-1} - G(X_\alpha)] + (I_p - B^{s\dagger}B^s)Z, \quad (27)$$

with U being found similarly as in (18) and

$$L_\alpha L_\alpha^T = X_\alpha - \Phi(\alpha, X_\alpha), \qquad T_\alpha T_\alpha^T = \overline{b^2C^sX_\alpha C^{s^T}} + \overline{b^2}\Sigma_w. \quad (28)$$

Since in inequality (25), α s can completely be absorbed into the A_k^s s, the system theoretic properties sufficient to lead to (25) will change to

the m.s. stabilizability of $(\alpha A_k^{s'}, b_k C_k^{s^T})$ and the m.s. observability of $(\alpha A_k^{s^T}, \Sigma_v^{1/2})$, which will be shown to be equivalent to the m.s. observability of $(A_k^{s^T}, \Sigma_v^{1/2})$. Since the new observability Grammian satisfies

$$\sum_{k=0}^{\ell} \alpha^{2k}\mathscr{A}^k(\Sigma_v) = [\Sigma_v^{1/2}, \alpha[\mathscr{A}(\Sigma_v)]^{1/2}, \ldots, \alpha^{\ell}[\mathscr{A}^{\ell}(\Sigma_v)]^{1/2} \times \begin{bmatrix} \Sigma_v^{T/2} \\ \alpha[\mathscr{A}(\Sigma_v)]^{T/2} \\ \vdots \\ \alpha^{\ell}[\mathscr{A}^{\ell}(\Sigma_v)]^{T/2} \end{bmatrix} \triangleq \xi_\alpha \xi_\alpha^T \quad (29)$$

the m.s. observability of $(\alpha A_k^{s^T}, \Sigma_v^{1/2})$ is equivalent to the full rank property of ξ_α. But

$$\begin{aligned}\text{rank}(\xi_\alpha) &= \text{rank}([\Sigma_v^{1/2}, [\mathscr{A}(\Sigma_v)]^{1/2}, \ldots, [\mathscr{A}^{\ell}(\Sigma_v)]^{1/2}] \times \text{diag}[I, \alpha I, \ldots, \alpha^{2} I]) \\ &= \text{rank}(\xi), \quad (30)\end{aligned}$$

with $\xi = \xi_\alpha$ for $\alpha = 1$, therefore m.s. observability of $(A_k^{s^T}, \Sigma_v^{1/2})$ and $(\alpha A_k^{s^T}, \Sigma_v^{1/2})$, $\alpha > 1$ are equivalent.

Equation (26) will also lead to

$$(I - B^s B^{s^\prime}) \times [X_\alpha - \alpha^2\overline{(A^s + bB^sK_\alpha)X_\alpha(A^s + bB^sK_\alpha)^T} - \Sigma_v] \times (I - B^sB^{s^\prime}) = 0 \quad (31)$$

similar to the development when $\alpha = 1$, so using Lemma 1 again, the existence of an $X_\alpha > 0$ is implied by the m.s. stabilizability of $(\alpha A_k^s, b_k B^s)$ pair.

If, for example, $\Sigma_v > 0$ or $(A_k^{s^T}, \Sigma_v^{1/2})$ is m.s. observable (please see the comment below Theorem 1) and the assignability conditions are satisfied for some $X_\alpha > 0$, then by Lemma 1, the operator $\mathscr{A}_\alpha^c : R^{n\times n} \rightarrow R^{n \times n}$ defined by

$$\mathscr{A}_\alpha^c(X_\alpha) = \alpha^2\overline{(A^s + bB^sK_\alpha C^s)X_\alpha(A^s + bB^sK_\alpha C^s)^T} \quad (32)$$

is a stable one or the spectrum of the operator $\alpha^{-2}\mathscr{A}_\alpha^c(\cdot)$ is inside a disc of radius $\alpha^{-2} < 1$, which will be called a m.s. stability margin of $1 - \alpha^{-2}$. If the covariance controller K_α based on the equation (24) is used in system (1), the actual second moment satisfies

$$X = \overline{(A^s + bB^sK_\alpha C^s)X(A^s + bB^sK_\alpha C^s)^T} + \overline{b^2}B^sK_\alpha\Sigma_wK_\alpha^TB^{s^T} + \Sigma_v. \quad (33)$$

Let us now see how this actual covariance is related to the design value X_α. Defining $X_d \triangleq X_\alpha - X$, we have from equations (24) and (33):

$$\begin{aligned}X_d &= \overline{(A^s + bB^sK_\alpha C^s)X_d(A^s + bB^sK_\alpha C^s)^T} \\ &+ (\alpha^2 - 1)[\overline{(A^s + bB^sK_\alpha C^s)X_\alpha(A^s + bB^sK_\alpha C^s)^T} \\ &+ \overline{b^2}B^sK_\alpha\Sigma_wK_\alpha^TB^{s^T}]. \quad (34)\end{aligned}$$

Since K_α is m.s. stabilizing with a stability margin $1 - \alpha^{-2}$, it is m.s. stabilizing. Also $\alpha^2 > 1$. Therefore, the right-most term above is non-negative definite. Hence, it follows from Lemma 1 that there exists a solution $X_d \geq 0$ to (34), which gives $X \leq X_\alpha$ or the true assigned covariance can be made to satisfy upper bounds included in X_α.

Theorem 2. Suppose that $(A_k^{s^T}, \Sigma_v^{1/2})$ is m.s. observable. Then the control gain (27)–(28) will result in a m.s. stability margin $1 - \alpha^{-2}$ as expressed by equation (32) and will satisfy the second moment upper bound $X \leq X_\alpha$ if and only if (25) holds with $X_\alpha - \Phi(\alpha, X_\alpha)$ having rank not exceeding p for some $\alpha > 1$ and (26) is true.

Now let us see how this extra degree of stability helps in the presence of parameter perturbations. Suppose that true parameters statistics are not known for A_k^s, b_k and C_k^s and we estimate the second moment operators $\overline{A_d^s(\cdot)A_d^{s^T}}$, $\overline{b_d^2}$, $\overline{b_d^2C_d^s(\cdot)C_d^{s^T}}$, and $\overline{b_dA_d^s(\cdot)C_d^{s^T}}$. This obviously encompasses deterministic as well as stochastic model uncertainties. We did not include any robustness results related to erroneous modeling of additive noise effects because such errors will not result in the destabilization of the system. In the choice of the design values for the parameter statistics, it is of utmost importance to make a selection in such a way as to enhance the satisfaction of the assignability conditions. Suppose that we would like to obtain a stable closed-loop system operator $\mathscr{A}_\beta^c$ for some $\beta > 1$. This means that under the proper assignability conditions, there exists a unique positive definite solution X_β to

$$X_\beta = \beta^2\overline{(A_d^s + b_dB^sK_\beta C_d^s)X_\beta(A_d^s + b_dB^sK_\beta C_d^s)^T} + \beta^2\overline{b_d^2}B^sK_\beta\Sigma_wK_\beta^TB^{s^T} + \Sigma_v, \quad (35)$$

which forms the basis of our robust control design. We shall use a Martingale (stochastic Lyapunov) type analysis to determine how much deviation from the true model statistics is possible without causing instability.

By using Lemma 2 for equation (35), we can see that there exists a positive definite solution to

$$X_\beta^* = \beta^2\overline{(A_d^s + b_dB^sK_\beta C_d^s)^T X_\beta^*(A_d^s + b_dB^sK_\beta C_d^s)} + \beta^2\overline{\beta_d^2}B^sK_\beta\Sigma_wK_\beta^TB^{s^T} + \Sigma_v, \quad (36)$$

where we will assume for simplicity that $\Sigma_v > 0$. Define the function $v_k = x_k^{s^T} X_\beta^* x_k^s$, which will serve as an auxiliary Lyapunov function. Define the difference

$$\mathscr{L}_k \triangleq E\{v_{k+1}/x_k^s, x_{k-1}^s, \ldots\} - v_k \quad (37)$$

which from (6) gives

$$\mathscr{L}_k = x_k^{s^T}[\overline{(A_d^c+\Delta^c)^T X_\beta^*(A_d^c+\Delta^c)} - X_\beta^*]x_k^s + \rho, \quad (38)$$

where A_{dk}^c, Δ_k^c, and ρ are defined by

$$A_{dk}^c = A_{dk}^s + b_{dk}B^s K_\beta C_{dk}^s \quad (39)$$

$$\Delta_k^c = A_k^s - A_{dk}^s + b_k B^s K_\beta C_k^s - b_{dk} B^s K_\beta C_{dk}^s \quad (40)$$

$$\rho = \overline{v^T X_\beta^* v} + \overline{b^2}\,\overline{w^T K_\beta^T B^{s^T} X_\beta^* B^s K_\beta w} > 0. \quad (41)$$

Since for any $\alpha > 0$, it is true that

$$\overline{A_d^{c^T} X_\beta^* \Delta^c} + \overline{\Delta^{c^T} X_\beta^* A_d^c} \le \alpha \overline{A_d^{c^T} X_\beta^* A_d^c} + \alpha^{-1}\overline{\Delta^{c^T} X_\beta^* \Delta^c} \quad (42)$$

letting $\alpha = \beta^2 - 1$, and substituting from (36), we have

$$\mathscr{L}_k \le x_k^{s^T}\left[\frac{\beta^2}{\beta^2-1}\overline{\Delta^{c^T} X_\beta^* \Delta^c} - \beta^2 \overline{b_d^2} B^s K_\beta \Sigma_w K_\beta^T B^{s^T} - \Sigma_v\right]x_k^s + \rho. \quad (43)$$

For m.s. and a.s. boundedness of the perturbed system state, it is sufficient that the expression in the square brackets be negative definite (Yaz, 1988a). That gives us the following result:

Theorem 3. Suppose that the assignability conditions of Theorem 2 (with α replaced by β) are met, and that a controller based on robustified design equation (35) is given. A sufficient condition to maintain the m.s. and a.s. boundedness of the system state for the deterministic and stochastic parameter perturbations defined above is that the total perturbation term defined by (40) satisfies the second moment bound

$$\overline{\Delta^{c^T} X_\beta^* \Delta^c} < (\beta^2 - 1)\overline{b_d^2} B^s K_\beta \Sigma_w K_\beta^T B^{s^T} + \frac{\beta^2-1}{\beta^2}\Sigma_v. \quad (44)$$

DYNAMIC OUTPUT FEEDBACK

(i) *Assignability and covariance assignment*

In this case, we put a further restriction on the input matrix by making it time-invariant, $B_k^s = B^s$. Let us use an r-dimensional linear time-invariant dynamic compensator

$$\begin{aligned} x_{k+1}^c &= A^c x_k^c + B^c y_k \\ u_k &= C^c x_k^c + D^c y_k \end{aligned} \quad (45)$$

that results in the closed-loop system

$$x_{k+1} = A_k^{CL} x_k + B^{CL}\zeta_k, \quad (46)$$

where

$$x_k = \begin{bmatrix} x_k^s \\ x_k^c \end{bmatrix} \in R^{n+r}, \quad A_k^{CL} = \begin{bmatrix} A_k^s + B^s D^c C_k^s & B^s C^c \\ B^c C_k^s & A^c \end{bmatrix} \quad (47)$$

$$B^{CL} = \begin{bmatrix} I_n & B^s D^c \\ 0 & B^c \end{bmatrix}, \quad \zeta_k = \begin{bmatrix} v_k \\ w_k \end{bmatrix}.$$

The second moment of the composite state is defined by

$$X \triangleq \lim_{k\to\infty} E\left\{\begin{bmatrix} x_k^s \\ x_k^c \end{bmatrix}[x_k^{s^T}\ x_k^{c^T}]\right\} = \begin{bmatrix} X^s & X^{sc} \\ (X^{sc})^T & X^c \end{bmatrix} \quad (48)$$

which satisfies

$$X = \overline{A^{CL} X (A^{CL})^T} + B^{CL} V (B^{CL})^T \quad (49)$$

if the controlled system is m.s. stabilized by the compensator (45), where

$$V \triangleq \begin{bmatrix} \Sigma_v & 0 \\ 0 & \Sigma_w \end{bmatrix}. \quad (50)$$

Expanding (49), we obtain

$$X = \overline{(A+BKC)X(A+BKC)^T} + (\mathrm{I}_{11} + \mathrm{BKI}_{12})\mathrm{V}(\mathrm{I}_{11} + \mathrm{BKI}_{12})^{\mathrm{T}} \quad (51)$$

with

$$A_k = \begin{bmatrix} A_k^s & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B^s & 0 \\ 0 & I_r \end{bmatrix}, \quad K = \begin{bmatrix} D^c & C^c \\ B^c & A^c \end{bmatrix} \quad (52)$$

$$C_k = \begin{bmatrix} C_k^s & 0 \\ 0 & I_r \end{bmatrix}, \quad I_{11} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad I_{12} = \begin{bmatrix} 0 & I_p \\ 0 & 0 \end{bmatrix}.$$

Upon rearrangement and simplification ($I_{12}VI_{11} = 0$), we obtain

$$X - \overline{AXA^T} - I_{11}VI_{11} = BK\overline{CXA^T} + \overline{AXC^T}K^TB^T + BK(\overline{CXC^T} + I_{12}VI_{12}^T)K^TB^T. \quad (53)$$

Completing the square on the right side gives

$$\begin{aligned} X - \overline{AXA^T} - I_{11}VI_{11} &= [BK + G(X)](\overline{CXC^T} + I_{12}VI_{12}^T)[BK + G(X)]^T \\ &\quad - G(X)(\overline{CXC^T} + I_{12}VI_{12}^T)G^T(X) \end{aligned} \quad (54)$$

with

$$G(X) = \overline{AXC^T}(\overline{CXC^T} + I_{12}VI_{12}^T)^{-1}. \quad (55)$$

The positive definiteness of the expression in the parentheses in (53) will guarantee the existence of the inverse. This expression is

$$\overline{CXC^T} + I_{12}VI_{12}^T = \begin{bmatrix} \overline{C^s X^s C^{s^T}} + \Sigma_w & \overline{C^s} X^{sc} \\ (X^{sc})^T \overline{C^{s^T}} & X^c \end{bmatrix}. \quad (56)$$

Note that our positive definite choice of X,

$$X = \begin{bmatrix} X^s & X^{sc} \\ (X^{sc})^T & X^c \end{bmatrix} > 0 \tag{57}$$

implies that $X^c > 0$ and $X^s - X^{sc}(X^c)^{-1}(X^{sc})^T > 0$ (Horn and Johnson, 1985). Hence the positive definiteness of (56) is guaranteed if $X^c > 0$ and $\overline{C^s X^s C^{s^T}} - \overline{C^s} X^{sc}(X^c)^{-1}(X^{sc})^T \overline{C^{s^T}} + \Sigma_w > 0$. But $\Sigma_w > 0$ is sufficient to make (56) positive definite, due to the fact that

$$\begin{aligned}&\overline{C^s X^s C^{s^T}} - \overline{C^s} X^{sc}(X^c)^{-1}(X^{sc})^T \overline{C^{s^T}} + \Sigma_w \\ &\quad = \overline{\tilde{C}^s X \tilde{C}^{s^T}} + \overline{C^s}[X^s - X^{sc}(X^c)^{-1}(X^{sc})^T]\overline{C^{s^T}} + \Sigma_w\end{aligned} \tag{58}$$

with $\tilde{C}^s_k \triangleq C^s_k - \overline{C^s}$. Therefore, the inverse in (56) will exist if, e.g. $X > 0$ is chosen and $\Sigma_w > 0$.

Rearranging equation (54) and substituting from (55) gives

$$\begin{aligned}&X - \overline{AXA^T} - I_{11}VI_{11} \\ &\qquad + \overline{AXC^T}(\overline{CXC^T} + I_{12}VI_{12}^T)^{-1}\overline{CXA^T} \\ &\quad = [BK + G(X)](\overline{CXC^T} + I_{12}VI_{12}^T) \\ &\qquad\qquad \times [BK + G(X)]^T.\end{aligned} \tag{59}$$

We require that the second moment X is chosen such that the left side of (59) is non-negative definite and of rank not exceeding $p + r$, because the right side has these properties. Under these conditions, (59) is rewritten as

$$LL^T = [BK + G(X)]TT^T[BK + G(X)]^T \tag{60}$$

with the obvious definitions of $L \in R^{(n+r)\times(p+r)}$ and $T \in R^{(p+r)\times(p+r)}$. So for an orthogonal $U \in R^{(p+r)\times(p+r)}$, we have

$$LU = [BK + G(X)]T. \tag{61}$$

It is shown above that $TT^T > 0$, so $\det[T] \neq 0$ and

$$BK = LUT^{-1} - G(X).$$

The solution for K exists if and only if

$$BB^+[LUT^{-1} - G(X)] = LUT^{-1} - G(X) \tag{62}$$

in which case, all solutions are given by

$$K = B^+[LUT^{-1} - G(X)] + (I_{m+r} - B^+B)Z \tag{63}$$

for an arbitrary $Z \in R^{(m+r)\times(p+r)}$. The necessary U matrix can be found from (61) as

$$U = \mathcal{V}_{1d}\,\text{block diag}\,[I_{p_d}, \mathcal{U}_d]\mathcal{V}_{2d}^T, \tag{64}$$

where the matrices above are found from the singular value decompositions $(I - BB^+)L(I - BB^+)L = \mathcal{W}_{1d}\,\text{block diag}\,[\Lambda_d, 0]\mathcal{V}_{1d}^T$, $(I - BB^+)G(X)T = \mathcal{W}_{1d}\,\text{block diag}\,[\Lambda_d, 0]\mathcal{V}_{2d}^T$ with $\mathcal{U}_d$ being row-orthonormal, and $\Lambda_d \in R^{p_d \times p_d}$ contains all the nonzero singular values. Condition (62) can also be simplified in a manner similar to the static output feedback case to yield

$$(I - BB^+)(X - \overline{AXA^T} - I_{11}VI_{11})(I - BB^+) = 0. \tag{65}$$

Let us summarize the previous development:

Theorem 4. A given steady state second moment $X \in R^{(n+r)\times(n+r)}$ with $X = X^T > 0$ is assignable by an r-dimensional dynamic compensator (45) to system (1) and (2) with $B^s_k = B^s$ and $\Sigma_w > 0$ if and only if X makes the right side of (59) non-negative definite with rank not exceeding $p + r$ and it satisfies equation (65). All compensators assigning X are given by (63) and (64).

(ii) *More on the dynamic assignability conditions*

Let us now expand on the assignability conditions starting with

$$\begin{aligned}X \geq\; &\overline{AXA^T} - \overline{AXC^T}(\overline{CXC^T} \\ &+ I_{12}VI_{12}^T)^{-1}\overline{CXA^T} + I_{11}VI_{11}.\end{aligned} \tag{66}$$

Upon substitution from (48), (50), and (52) and after a few manipulations involving the inverse of partitioned matrices, (66) yields

$$\begin{bmatrix} Y & X^{sc} \\ (X^{sc})^T & X^c \end{bmatrix} \geq 0, \tag{67}$$

where

$$\begin{aligned}Y = X^s &- \overline{\tilde{A}^s X^s \tilde{A}^{s^T}} - \bar{A}^s W \bar{A}^{s^T} - \Sigma_v \\ &+ (\overline{\tilde{A}^s X^s \tilde{C}^{s^T}} + \bar{A}^s W \bar{C}^{s^T}) \\ &\times [\overline{\tilde{C}^s X^s \tilde{C}^{s^T}} + \bar{C}^s W \bar{C}^{s^T} + \Sigma_w]^{-1} \\ &\times (\overline{\tilde{A}^s X^s \tilde{C}^{s^T}} + \bar{A}^s W \bar{C}^{s^T})^T\end{aligned} \tag{68}$$

and

$$W = X^s - X^{sc}(X^c)^{-1}(X^{sc})^T \tag{69}$$

is the Schur complement of X^s. Since $X > 0$ is chosen, both X^c and W will be positive definite. Therefore, in order to satisfy (67), we only need to have (Horn and Johnson, 1985):

$$Y - X^{sc}(X^c)^{-1}(X^{sc})^T \geq 0 \tag{70}$$

which simplifies to the Riccati matrix inequality in W with a cross-weighting term

$$\begin{aligned}W \geq\; &\bar{A}^s W \bar{A}^{s^T} - (\bar{A}^s W \bar{C}^{s^T} + \overline{\tilde{A}^s X^s \tilde{C}^{s^T}}) \\ &\times (\bar{C}^s W \bar{C}^{s^T} + \Sigma_w + \overline{\tilde{C}^s X^s \tilde{C}^{s^T}})^{-1} \\ &\times (\bar{A}^s W \bar{C}^{s^T} + \overline{\tilde{A}^s X^s \tilde{C}^{s^T}})^T + \Sigma_v + \overline{\tilde{A}^s X^s \tilde{A}^{s^T}}.\end{aligned} \tag{71}$$

First, note that

$$\begin{aligned}Q \triangleq\; &\Sigma_v + \overline{\tilde{A}^s X^s \tilde{A}^{s^T}} - \overline{\tilde{A}^s X^s \tilde{C}^{s^T}} \\ &\times [\overline{\tilde{C}^s X^s \tilde{C}^{s^T}} + \Sigma_w]^{-1}\overline{\tilde{C}^s X^s \tilde{A}^{s^T}}\end{aligned} \tag{72}$$

is non-negative definite, since this property of Q for $\Sigma_w > 0$ is equivalent to the same property of

the partitioned matrix (Horn and Johnson, 1985):

$$\begin{bmatrix} \Sigma_v + \overline{\bar{A}^s X^s \bar{A}^{s^T}} & \overline{\bar{A}^s X^s \bar{C}^{s^T}} \\ \overline{\bar{C}^s X^s \bar{A}^{s^T}} & \Sigma_w + \overline{\bar{C}^s X^s \bar{C}^{s^T}} \end{bmatrix} \geq 0 \quad (73)$$

which is obviously true since (73) can be written as

$$\begin{bmatrix} \Sigma_v & 0 \\ 0 & \Sigma_w \end{bmatrix} + E\left\{\begin{bmatrix} \bar{A}^s \\ \bar{C}^s \end{bmatrix} X^s [\bar{A}^{s^T} \bar{C}^{s^T}]\right\} \geq 0. \quad (74)$$

So, rewriting (71) as

$$W \geq F[W - W\bar{C}^{s^T}(\bar{C}^s W \bar{C}^{s^T} + R)^{-1}\bar{C}^s W]F^T + Q \quad (75)$$

with

$$F = \bar{A}^s - \overline{\bar{A}^s X^s \bar{C}^{s^T}} R^{-1} \overline{\bar{C}^s X^s \bar{A}^{s^T}}, \quad R = \Sigma_w + \overline{\bar{C} X^s \bar{C}^{s^T}}. \quad (76)$$

Note that the situation in (66) is quite different from that of (19) in the static output feedback case, since now the parameters F and Q are dependent on the covariance $X^s = W + X^{sc}(X^c)^{-1}(X^{sc})^T$ whose existence is sought. Therefore, we cannot use reachability and detectability properties to secure the existence of a solution to the non-negativity condition.

Now considering condition (65) and upon substitution, we see that it reduces to

$$\begin{bmatrix} (I - B^s B^{s^\dagger})(X^s - \overline{A^s X^s A^{s^T}} - \Sigma_v)(I - B^s B^{s^\dagger}) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (77)$$

so that it can be given the same interpretation as in the static output feedback case, i.e. it is implied by the m.s. stabilizability property of (A_k^s, B^s) pair.

(iii) *Robust design*

To robustify this dynamic compensator, we start with a modification of equation (51):

$$X_\alpha = \alpha^2 \overline{(A + BK_\alpha C)X_\alpha(A + BK_\alpha C)^T} + \alpha^2(I_{11} + BK_\alpha I_{12})V(I_{11} + BK_\alpha I_{12})^T \quad (78)$$

for an $X_\alpha = X_\alpha^T > 0$ that gives rise to the non-negativity condition:

$$Z_\alpha - \alpha^2 \overline{AX_\alpha A^T} - \alpha^2 I_{11} V I_{11} + \alpha^2 \overline{AX_\alpha C^T}(\overline{CX_\alpha C^T} + I_{12} V I_{12}^T)^{-1} \overline{CX_\alpha A^T} \geq 0 \quad (79)$$

with the same rank condition. The consistency condition becomes

$$(I - BB^\dagger)(X_\alpha - \alpha^2 \overline{AX_\alpha A^T} - \alpha^2 I_{11} V I_{11}) \times (I - BB^\dagger) = 0. \quad (80)$$

Suppose that these assignability conditions are satisfied, then the control gain becomes

$$K_\alpha = B^\dagger[L_\alpha U T_\alpha^{-1} - G(X_\alpha)] + [I_{m+r} - B^\dagger B]Z \quad (81)$$

with $L_\alpha L_\alpha^T$ being equal to the left side of inequality (79) and

$$T_\alpha T_\alpha^T = \alpha^2[\overline{CX_\alpha C^T} + I_{12} V I_{12}^T] \quad (82)$$

the orthogonal matrix U being found as in (64).

Again, denoting the difference between the design and actually assigned covariances by $X_d = X_\alpha - X$, we find from (51) and (78) that

$$X_d = \overline{(A + BK_\alpha C)X_d(A + BK_\alpha C)^T} + (\alpha^2 - 1)[\overline{(A + BK_\alpha C)X_\alpha(A + BK_\alpha C)^T} + (I_{11} + BK_\alpha I_{12})V(I_{11} + BK_\alpha I_{12})^T]. \quad (83)$$

Since $\alpha^2 > 1$, the right-most term is non-negative definite and since K_α stabilizes the system with an extra degree of m.s. stability, the closed loop system operator is a stable one. Therefore, by Lemma 1, there exists a solution $X_d \geq 0$ to (83) which implies that an upper bound $X_\alpha \geq X$ on the assigned covariance can be achieved.

Theorem 5. The necessary and sufficient conditions for the assignability of $X_\alpha > 0$ to (78) are that (79) and (80) hold and that the left side of (79) should not exceed $(p + r)$ in rank. In this case, the control specified by (81)–(82) will assign a second moment bound on the closed-loop covariance (51) that satisfies equation (78) and the closed-loop system will have m.s. stability margin of $1 - \alpha^{-2}$ and the a.s. boundedness of the closed-loop system will be guaranteed if $(A_k + BK_\alpha C_k, (I_{11} + BK_\alpha I_{12})V^{1/2})$ is m.s. observable.

The last statement of the above theorem follows from Lemma 1 and the discussion on robust design for static output feedback controllers.

Let us now use this robust design to accommodate erroneous or unmodeled parameter statistics. Assume again that the available model has the operators $\overline{A_d(\cdot)A_d^T}$, $\overline{C_d(\cdot)A_d^T}$, and $\overline{C_d(\cdot)C_d^T}$ instead of the true moments of A_k and C_k in (52). Therefore, we base our design on

$$X_\beta = \beta^2 \overline{(A_d + BK_\beta C_d)X_\beta(A_d + BK_\beta C_d)^T} + \beta^2(I_{11} + BK_\beta I_{12})V(I_{11} + BK_\beta I_{12})^T \quad (84)$$

assuming that the modified assignability conditions for $X_\beta = X_\beta^T > 0$ are met. Again, we see that the use of Lemma 2 allows the existence of a positive definite X_β^* satisfying

$$X_\beta^* = \beta^2 \overline{(A_d + BK_\beta C_d)^T X_\beta^*(A_d + BK_\beta C_d)} + \beta^2(I_{11} + BK_\beta I_{12})V(I_{11} + BK_\beta I_{12})^T. \quad (85)$$

A similar analysis to the static output case yields,

$$\mathcal{L}_k = x_k^T[\overline{(A_d^c + \Delta^c)^T X_\beta^*(A_d^c + \Delta^c)} - X_\beta^*]x_k + \mu \quad (86)$$

with A_{dk}^c, Δ_k^c, μ defined by

$$A_{dk}^c = A_{dk} + BK_\beta C_{dk} \quad (87)$$

$$\Delta_k^c = A_k - A_{dk} + BK_\beta(C_k - C_{dk}) \quad (88)$$

$$\mu = \overline{\xi^T(I_{11} + BK_\beta I_{12})^T X_\beta^*(I_{11} + BK_\beta I_{12}\xi)} > 0. \quad (89)$$

Again using the fact

$$\overline{A_d^{c^T} X_\beta^* \Delta^c} + \overline{\Delta^{c^T} X_\beta^* A_d^c} \le (\beta^2 - 1)\overline{A_d^{c^T} X_\beta^* A_d^c} + (\beta^2 - 1)^{-1}\overline{\Delta^{c^T} X_\beta^* \Delta^c} \quad (90)$$

and substituting from (85), $\mathcal{L}_k \le -x_k^T M x_k + \mu$ for an $M > 0$ if

$$\overline{\Delta^{c^T} X_\beta^* \Delta^c} < (\beta^2 - 1)(I_{11} + BK_\beta I_{12})V(I_{11} + BK_\beta I_{12})^T. \quad (91)$$

The right side of inequality (91) can be made positive definite if $\Sigma_v > 0$, $\Sigma_w > 0$ and B^c is full row rank with the dimension of the controller satisfying $r \le p$. This can be seen easily from

$$\begin{aligned}\det[(I_{11} + BK_\beta I_{12})V(I_{11} + BK_\beta I_{12})^T] &= \det[\Sigma_v + B^s D^c \Sigma_w D^{c^T} B^{s^T}] \\ &\quad \times \det[B^c \Sigma_w B^{c^T} - B^c \Sigma_w D^{c^T} B^{s^T} \\ &\quad \times (\Sigma_v + B^s D^c \Sigma_w D^{c^T} B^{s^T})^{-1} B^s D^c \Sigma_w B^{c^T}] \\ &= \det[\Sigma_v + B^s D^c \Sigma_w D^{c^T} B^{s^T}] \\ &\quad \times \det[(B^c(\Sigma_w^{-1} + D^{c^T} B^{s^T} \Sigma_v^{-1} B^s D^c)^{-1} B^{c^T}]\end{aligned} \quad (92)$$

which follows from (52) and the matrix inversion lemma. If the full rank property does not hold, then one can add diag $[0, \varepsilon I_r]$, $0 < \varepsilon \ll 1$ to the covariance equation to make

$$(I_{11} + BK_\beta I_{12})V(I_{11} + BK_\beta I_{12})^T + \text{diag}[0, \varepsilon I_r] > 0 \quad (93)$$

if $\Sigma_w > 0$. This changes the derivation of the controller in a trivial way and it can also easily be shown that this assigns a yet higher value to the covariance (in the positive semidefinite sense).

Theorem 6. Suppose that the assignability conditions of Theorem 5 are met with α replaced with $\beta > 0$, and the corresponding K_β is given from (81), where A_d, C_d are given. Let the errors in A_d, C_d be characterized by (88), (91) where X_β^* is defined by (85), and $\Sigma_v > 0$, $\Sigma_w > 0$, $B^c B^{c^T} > 0$ [or, if not, design is accomplished with diag $(0, \varepsilon I_p)$, $0 < \epsilon \ll 1$, added to equation (84)]. Then the m.s. and a.s. boundedness of the system state can be maintained if the perturbations satisfy the second moment bound (91).

EXAMPLE

Let us consider the simply supported beam example of Skelton *et al.* (1991). The discretization of equations with a sampling period of 0.05 s will result in

$$\bar{A}^s = \begin{bmatrix} 0.9988 & 0.0500 & 0 & 0 \\ -0.0500 & 0.9983 & 0 & 0 \\ 0 & 0 & 0.9801 & 0.0496 \\ 0 & 0 & -0.7939 & 0.9781 \end{bmatrix},$$

$$\bar{B}^s = \begin{bmatrix} 0.0007 & -0.0013 \\ 0.0294 & -0.0500 \\ 0.0012 & 0.0025 \\ 0.0472 & 0.0992 \end{bmatrix}$$

$$\Sigma_v = \begin{bmatrix} 0.0500 & 0 & 0 & 0 \\ 0 & 0.0500 & 0 & 0 \\ 0 & 0 & 0.0494 & -0.0186 \\ 0 & 0 & -0.0186 & 0.0599 \end{bmatrix},$$

$$\bar{C}^s = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0.9511 & 0 & -0.5878 & 0 \end{bmatrix}.$$

We assume that there is a measurement noise of zero mean and covariance $\Sigma_w = 10^{-4} I_2$.

Also, structured parameter perturbations are modeled as white noises such that

$$\begin{aligned} A_k(\omega) &= \bar{A}^s + a_k(\omega)A_1, \\ C_k(\omega) &= \bar{C}^s + c_k(\omega)C_1, \end{aligned} \quad (94)$$

where a_k and c_k are both standard scalar white noise sequences and

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.0200 & 0.0065 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.425 & 0.0275 \end{bmatrix}, \quad (95)$$

$$C_1 = 10^{-3}\begin{bmatrix} 0.1000 & 0 & 0.02000 & 0 \\ 0.1000 & 0 & 0.0500 & 0 \end{bmatrix}.$$

We can view (94) and (95) as describing the maximum amount of parameter perturbations for this system, so if we design our controller based on these values, then the closed-loop system will remain stable in the senses described before, for all stochastic perturbations with smaller variances.

Let us suppose that we are allowed to use a third order controller and would like to assign a covariance so that the following mean square values of the output variables dictated by the pointing accuracy requirements are satisfied:

$$\lim_{k\to\infty} E\{[(y_k^{(1)})^2, (y_k^{(2)})^2]\} = [6.7, 2.0].$$

By using the simplified assignability conditions (75)–(77), an assignable closed-loop (composite)

state covariance which satisfies these conditions is given by:

$$X = \begin{bmatrix} 1.7534 & -0.5002 & 0.0106 & 0.5196 & -2.2410 & 1.4071 & -7.7848 \\ -0.5002 & 3.0338 & -0.5213 & -0.1013 & 2.4778 & -2.1106 & 9.0291 \\ 0.0106 & -0.5213 & 1.2282 & -0.5017 & -2.5920 & -0.2600 & -3.3558 \\ 0.5196 & -0.1013 & -0.5017 & 18.2277 & 4.3542 & -5.0918 & 15.7135 \\ -2.2410 & 2.4778 & -2.5920 & 4.3542 & 9.8070 & -3.3674 & 23.5482 \\ 1.4071 & -2.1106 & -0.2600 & -5.0918 & -3.3674 & 5.3178 & -17.4382 \\ -7.7848 & 9.0291 & -3.3558 & 15.7135 & 23.5482 & -17.4382 & 78.8709 \end{bmatrix}$$

with a spectrum of {0.0486, 0.0375, 0.4183, 2.1232, 4.8230, 15.3486, 95.4397}. This covariance is found by a recursion that involves solving an algebraic Riccati equation at every step. Since we do not yet have a theoretical convergence result for this algorithm, it is not included here.

We chose $\bar{U} = I$, $Z = 0$ for simplicity which gives the gain matrix, from (63) and (64)

$$K = \begin{bmatrix} -29.1605 & -25.8431 & -29.7540 & 33.0330 & 2.2520 \\ -9.3691 & 22.6058 & -73.9622 & 80.0310 & 38.3695 \\ -1.7749 & -0.3840 & -0.7218 & 0.3731 & 0.0873 \\ 0.8911 & 1.5140 & 0.2140 & -0.1089 & 0.2633 \\ -4.1650 & -4.9793 & -0.0281 & -0.0427 & -0.9907 \end{bmatrix}$$

or

$$A_c = \begin{bmatrix} -0.7218 & 0.3731 & 0.0873 \\ 0.2140 & -0.1089 & 0.2633 \\ -0.0281 & -0.0427 & -0.9907 \end{bmatrix},$$

$$B_c = \begin{bmatrix} -1.7749 & -0.3840 \\ 0.8911 & 1.5140 \\ -4.1650 & -4.9793 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -29.7540 & 33.0330 & 2.2520 \\ -73.9622 & 80.0310 & 38.3695 \end{bmatrix},$$

$$D_c = \begin{bmatrix} -29.1605 & -25.8431 \\ -9.3691 & 22.6058 \end{bmatrix}$$

which results in a closed-loop system which is a m.s. and a.s. stable with the desired steady state covariance. Since this design guarantees stability for the maximum possible perturbations defined in (93)–(94), we do not need to further robustify this controller.

CONCLUSION

This work contains a characterization of all covariances that a discrete stochastic parameter system may have, and parametrizes all static and fixed-order linear dynamic output feedback controllers that assign a desired covariance. Robustification of the controllers is achieved for use in an erroneously modeled system or in one with deterministic and random parameter perturbations. Although this work implicitly achieves a parametrization of all mean-square stabilizing controllers, which is of prime theoretical importance, at the implementation stage, we have had several difficulties. If a desired covariance can be found from maximum mean-square values of the regulated state variables, but does not satisfy the assignability conditions, then one should be able to find an assignable covariance that is close in some sense to the desired covariance. This implementation issue and others like the numerical generation of all assignable covariances for a given system have been the subject of recent research (Grigoriadis and Skelton, 1992) and positive results will be made available soon.

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